The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring semester 2015-2016**

**Advanced Operations Research** **2 Hours**

*Attempt all FOUR questions.*

**1** (i) Use the two-phase method to find the optimal solution for the following linear programming problem:

$$\max \quad z = -4x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + x_2 \leq 6,$$
$$3x_1 + x_2 \geq 3,$$
$$2x_1 - 2x_2 \leq 1.$$

Clearly state your final solution. Hint: not counting the preprocessing step, you need three tableaux in phase 1. ***(15 marks)***

(ii) Starting from the definition of Lagrangian dual functions, derive the dual linear programming problem of the above problem. ***(10 marks)***

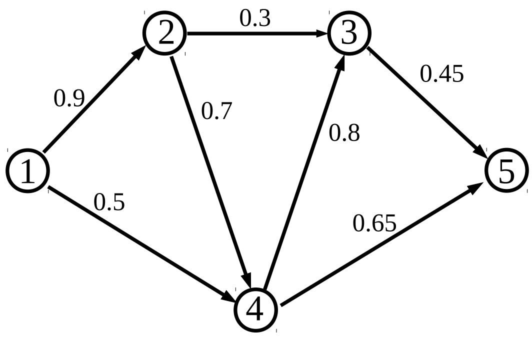


Figure 1: The network for question 2(i).

**2** (i) Figure 1 provides the communication network between two stations, 1 and 5. The probability that a link in the network will operate without failure is shown on each arc connecting two stations. Messages are sent from station 1 to station 5. The objective is to determine the route that will maximise the probability of a successful transmission. Using the shortest-route model to model the problem, find its formulation in the form of a mixed integer-linear programming problem. **Find the formulation only; do NOT try to solve it**.

Hint: The probability to have a successful transmission over a route is the product of the probabilities for all the arcs on the route. Your objective should be maximizing the logarithm of the former, which is the sum of the logarithm of the probabilities of individual arcs. ***(9 marks)***

**2** (continued)

(ii) A manufacturer is using two ingredients and a filler to produce a mixture for a dietary supplement tablet. The mixture may contain three types of nutrients: A, B and C. Several requirements have to be met:

- The mixture has to meet minimum requirements for *at least two* of the three nutrients. In one kilogram of the mixture, the minimum requirement is 75 grams for A, 50 grams for B, and 20 grams for C. If two of the three requirements are satisfied, there is no requirement on the third nutrient.
- A fixed set-up cost of £15 will be incurred if ingredient 2 is used.
- The filler, though it contributes to the weight of the mixture, does not have nutritional contents, and its cost can be neglected.
- The manufacturer will need to produce 100 kilograms of the mixture.

The nutrient contents (unit: gram per kilogram of the ingredient) and the costs (unit: pound per kilogram of the ingredient) of each ingredient are given in the following table:

| | A (g/kg) | B (g/kg) | C (g/kg) | Cost (£/kg) |
|---|---|---|---|---|
| Ingredient 1 | 100 | 80 | 40 | 4 |
| Ingredient 2 | 75 | 150 | 20 | 6 |

Formulate the mixed integer linear programming problem with which the manufacturer can solve for the optimal amounts of the two ingredient in **1 kilogram** of the mixture. **Find the formulation only; do NOT try to solve it**. ***(16 marks)***

**3** We consider a minimisation linear programming problem:

$$\min \ z = c^T x, \quad \text{subject to} \quad Ax \leq b \text{ and } x \geq 0.$$

We use $B$ to denote the optimal basic matrix for the problem, and $c_B$ the corresponding cost coefficient vector. It is given that the dual problem is:

$$\max \ v = -b^T y, \quad \text{subject to} \quad -A^T y \leq c \text{ and } y \geq 0,$$

and we let $y^+ = -B^{-T} c_B$.

(i) Show that $y^+$ is a feasible optimal solution for the dual problem, and that the strong duality condition is satisfied. ***(15 marks)***

(ii) Show that $y^+$ is also the shadow costs for the primal problem. ***(3 marks)***

(iii) Write down the **two** sets of complementary slackness conditions corresponding to this pair of primal-dual problems, and give a brief interpretation of their implications in terms of the shadow costs and reduced costs of the primal problem. ***(7 marks)***

**4** Cartoy is a manufacturer that produces two types of toy cars. The following information is known:

- Car 1 is sold at £25. It requires one hour's work from worker 1 and two hours' work from worker 2, and needs raw material worth £5.
- Car 2 is sold at £22. It requires two hours' work from worker 1 and one hour's work from worker 2, and needs raw material worth £4.
- Worker 1 can work up to 40 hours per week and is paid £5 per hour. Worker 2 can work up to 50 hours per week and is paid £6 per hour. As a consequence, the profit of producing a Car 1 is £3, and a Car 2 is £2.
- There is unlimited supply for raw material.

Let $x_1$ and $x_2$ be the numbers of car 1 and car 2 produced, respectively, and $x_3$ and $x_4$ be the slack variables relating to the maximum working hours of worker 1 and worker 2, respectively. One can formulate the linear programming problem as follows:

$$\max \quad z = 3x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 40 \\ 2x_1 + x_2 &\leq 50 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solving the problem with the simplex method, we find the following optimal tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Solution |
|---|---|---|---|---|---|
| $z$ | 0 | 0 | 1/3 | 4/3 | 80 |
| $x_1$ | 1 | 0 | $-1/3$ | 2/3 | 20 |
| $x_2$ | 0 | 1 | 2/3 | $-1/3$ | 10 |

(i) Using the information given in the optimal tableau, find the optimal solutions for the decision variables and the cost function, and the optimal solutions for the dual variables. ***(3 marks)***

(ii) Determine the range of the prices for car 1 for which the current basis remains optimal. ***(8 marks)***

(iii) If worker 2 can work only up to 15 hours per week due to medical conditions, would the current basis remain feasible and optimal? If not, find the new optimal feasible solution using the dual simplex method. Clearly state your new optimal solution. ***(16 marks)***

## End of Question Paper