Data provided: Formula sheet

The University Of Sheffield.

MAS424

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2015–2016

Differential Equations: Advanced — **2 hours**

Attempt all questions.

1 Consider the system of equations

$$\frac{dx}{dt} = -y + xF(r), \qquad \frac{dy}{dt} = x + yF(r). \tag{1}$$

where $r^2 = x^2 + y^2$, and $F(r)$ is a real-valued continuous function of r.

(i) Show that $x = 0$, $y = 0$ is the only critical point of the system. By considering the linearised system, show that this critical point is a spiral. What is the nature of the critical point if $F(0) > 0$? ***(8 marks)***

(ii) Use the variable substitution

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

to show that the system (1) can be written as

$$\frac{dr}{dt} = rF(r), \qquad \frac{d\theta}{dt} = 1.$$

Thus show that system (1) has a periodic solution for each value r_0 of r such that $F(r_0) = 0$. ***(5 marks)***

(iii) By considering a small perturbation, $r = r_0 + \delta$, about r_0 and using a Taylor expansion of function $F(r)$, show that this periodic solution is a stable limit cycle in the case $F'(r_0) < 0$, and it is an unstable limit cycle in the case $F'(r_0) > 0$. ***(5 marks)***

(iv) Find all limit cycles of the system

$$\frac{dx}{dt} = x - y - x\sqrt{x^2 + y^2}, \quad \frac{dy}{dt} = x + y - y\sqrt{x^2 + y^2},$$

and determine their stability. Sketch a phase portrait for the system. ***(7 marks)***

2 A model for the interaction of a predator v and its prey u us described by

$$\begin{aligned}\frac{du}{dt} &= Ru\left(1-\frac{u}{K}\right) - A_1\frac{u}{A_2+u}v\\ \frac{dv}{dt} &= -B_1v + B_2\frac{u}{A_2+u}v,\end{aligned}$$

where R, K, A_1, A_2, B_1 and B_2 are positive constants.

(i) Explain the biological meaning of each of the four terms in the model. ***(4 marks)***

(ii) Use the variable substitutions

$$x = \frac{u}{K}, \quad y = \frac{A_1}{KR}v, \quad T = Rt,$$

to show that the system can be written in the dimensionless form

$$\begin{aligned}\frac{dx}{dT} &= x(1-x) - \frac{x}{\theta+x}y\\ \frac{dy}{dT} &= -ay + c\frac{x}{\theta+x}y,\end{aligned}$$

and find expressions for θ, a and c in terms of the model parameters. ***(3 marks)***

(iii) Show that the model always has critical points at $(0,0)$ and $(1,0)$, and determine the nature of these critical points. ***(4 marks)***

(iv) Show that the model has a third critical point (x_*, y_*) if and only if $\theta < \dfrac{c-a}{a}$. Determine how the stability of the steady state (x_*, y_*) depends on the parameters θ, a and c. ***(5 marks)***

(v) Sketch the nullclines and typical phase portrait for

(a) $\theta > \dfrac{c-a}{a}$, (b) $\dfrac{c-a}{c+a} < \theta < \dfrac{c-a}{a}$, (c) $\theta < \dfrac{c-a}{c+a}$.

If the system is at the steady state $(1,0)$ and a small number of predators is introduced, describe briefly how the system will respond in each of cases (a)–(c). Show the corresponding trajectories on your phase portraits. ***(9 marks)***

3 The soil temperature, $u(x,t)$, at a depth x and time t can be modelled by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty,\ t > 0, \qquad (2)$$

where $D > 0$ is the thermal conductivity of the soil. The temperature at the surface is given by

$$u(0,t) = u_0 + u_1 \cos \omega t,$$

where u_0, u_1 and ω are constants, and $\omega > 0$.

(i) Sketch the form of the surface temperature. What is its periodicity? ***(3 marks)***

(ii) Assuming a solution of (2) of the form

$$u(x,t) = A + f(x)\cos(\omega t - \kappa x), \quad u(x,t) \to u_0 \text{ as } x \to \infty,$$

with $\kappa > 0$, determine the constants A and κ, and the function $f(x)$. ***(10 marks)***

(iii) The mean daily air temperature in Sheffield varies *annually* between a minimum of 4°C and a maximum of 18°C. Determine appropriate values of u_0, u_1 and ω for these annual variations. ***(3 marks)***

(iv) If $D = 2 \times 10^{-3}\,\text{cm}^2\text{s}^{-1}$, determine the depth X at which the mean temperature fluctuates by 1°C. If the maximum mean temperature occurs on 25 July, on which day of the year is the soil at depth X at its maximum temperature? ***(6 marks)***

(v) By what factor would you expect *daily* temperate fluctuations to be attenuated by at depth X? ***(3 marks)***

4 (i) The Euler-Lagrange equation corresponding to a functional $f(x, y(x), y'(x))$ is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}.$$

If f is independent of x, show that

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}.$$

(6 marks)

(ii) Consider two points A and B with coordinates $(-1, 1)$ and $(1, 1)$ respectively. Show that the functional

$$I = \int_A^B \frac{\sqrt{1 + y'^2}}{y}\,\mathrm{d}x$$

achieves an extremal value when $x^2 + y^2 = 2$. Sketch this path between A and B. ***(10 marks)***

(iii) Show that the value of I evaluated along this path is $I_0 = 2\ln(1 + \sqrt{2})$. ***(6 marks)***

(iv) By evaluating I along the straight line path between A and B, show that I_0 is a minimal value for I. ***(3 marks)***

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when} \;\; m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$