The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Optics and Symplectic Geometry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.

1 (i) Let S be a $2n \times 2n$ matrix (with real entries). Define what it means for S to be a *symplectic matrix.*

Now write S in block form as

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are $n \times n$ matrices. Show that S is symplectic if and only if all the following hold

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \qquad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \qquad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I.$$

(4 marks)

(ii) Suppose that $S \in Sp(2n)$ has a real eigenvalue λ with a corresponding eigenvector $Z \in \mathbb{R}^{2n}$. Show that JZ is an eigenvector for S^{T} and find the corresponding eigenvalue. **(4 marks)**

(iii) Let (V, ω) be a symplectic vector space of dimension $2n$. Define the concept of *symplectic basis* for (V, ω).

Now consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega)$. Let $e_1, \ldots, e_n$ be any orthonormal basis in $\mathbb{R}^n$. Prove that

$$(e_1, 0), \ldots, (e_n, 0), (0, e_1), \ldots, (0, e_n)$$

is a symplectic basis in $\mathbb{R}^{2n}$. **(8 marks)**

(iv) Take $X \in \mathbb{R}^3$ and $Y \in \mathbb{R}^3$ with $|X| = 1$ and $X \cdot Y = 0$. You are given that

$$V := \{(\xi, \eta) \in \mathbb{R}^6 \mid X \cdot \xi = 0, \quad X \cdot \eta + \xi \cdot Y = 0\}.$$

is a symplectic subspace of $(\mathbb{R}^6, \Omega)$.

(a) Let X, B_2, B_3 be an orthonormal basis for $\mathbb{R}^3$ containing X, and write Y with respect to this basis as $Y = y_2 B_2 + y_3 B_3$.

In terms of coordinates with respect to this basis, write down the conditions that $(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3$ belong to V.

For $(\xi, \eta), (\xi', \eta') \in V$, express $\Omega((\xi, \eta), (\xi', \eta'))$ in terms of these coordinates.

Find a symplectic basis for V in terms of X, B_2, B_3.

(b) Now assume that $Y = 0$. Show that the subspace

$$L = \{(\xi, \eta) \in V \mid \xi = \eta\}$$

is Lagrangian in V and find another Lagrangian subspace L' of V such that $V = L \oplus L'$, proving the stated properties.

(16 marks)

2 (i) Let W be a vector space of dimension k.

Define the *dual space* W^*, being sure to include definitions of the vector space operations. State, without proof, the dimension of W^*. **(6 marks)**

(ii) Let (V, ω) be a symplectic vector space of dimension $2n$ and let L_1 and L_2 be Lagrangian subspaces such that $V = L_1 \oplus L_2$.

Define a map $\Phi\colon L_1 \to L_2^*$ by $(\Phi(v_1))(v_2) = \omega(v_1, v_2)$ for $v_1 \in L_1$ and $v_2 \in L_2$. State why Φ takes values in L_2^* and show that it is a linear map.

Prove that Φ is an isomorphism of vector spaces, stating clearly any general result of linear algebra which you use. **(12 marks)**

3 (i) Snell's Law may be given as $n' \sin\theta' = n \sin\theta$ for refraction across a boundary between mediums with indexes of refraction n and n', where θ and θ' are the angles made by the rays with the normal to the boundary.

Derive Snell's Law in vector form so that it applies to rays in $\mathbb{R}^3$, including a simple diagram in your answer. Your answer should be in terms of unit vectors v and v' along the incoming and outgoing rays, a unit vector Σ normal to the boundary betwen the two regions, and the indexes of refraction n and n'. **(10 marks)**

(ii) (a) Calculate the matrix product

$$\begin{bmatrix} I & w'I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ M & I \end{bmatrix} \begin{bmatrix} I & wI \\ 0 & I \end{bmatrix}$$

where M is symmetric, and describe the optical situation which leads to finding the product of these matrices. **(12 marks)**

(b) Let $S \in Sp(4)$ be given in block form by $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Suppose that there are real numbers $w, w' > 0$ such that

$$A - I = w'C, \qquad D - I = wC.$$

Assuming that C is invertible, express B in terms of w, w' and C. **(5 marks)**

4 In this question each $\mathbb{R}^{2n}$ has the standard symplectic form Ω.

(a) Let W be an n–dimensional subspace of $\mathbb{R}^{2n}$. Take a basis of W and write the elements as the columns of a $2n \times n$ matrix which we write in block form as

$$\begin{bmatrix} M \\ N \end{bmatrix}$$

where M and N are $n \times n$ matrices.

Prove that W is Lagrangian if and only if $M^{\mathsf{T}}N$ is symmetric. **(4 marks)**

(b) Let $L \subseteq \mathbb{R}^{2n}$ be a Lagrangian subspace of $\mathbb{R}^{2n}$. Show that it is transverse to both $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$ if and only if it has a representation as in (a) of the form $\begin{bmatrix} M \\ I \end{bmatrix}$ with M invertible. **(8 marks)**

(c) Let L and L' be Lagrangian subspaces which are both transversal to $\mathbb{R}^n \times 0$ and $0 \times \mathbb{R}^n$. State without proof the theorem which gives criteria for the existence of $S \in Sp(2n)$ such that $S(\mathbb{R}^n \times 0) = \mathbb{R}^n \times 0$, $S(0 \times \mathbb{R}^n) = 0 \times \mathbb{R}^n$, and $S(L) = L'$. **(3 marks)**

(d) Let L be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$(1, 3, 5, 1, 0, 0), \quad (0, 1, 0, 1, -2, 1), \quad (0, 1, 2, 5, 0, -1).$$

Show that L is Lagrangian in $(\mathbb{R}^6, \Omega)$ and that it is transverse to both $\mathbb{R}^3 \times 0$ and $0 \times \mathbb{R}^3$. Represent L in the form

$$\begin{bmatrix} M \\ I \end{bmatrix}$$

as in (b) and determine the signature of M. **(8 marks)**

End of Question Paper