The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Galois Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper K denotes a subfield of $\mathbb{C}$ which contains $\mathbb{Q}$. All field extensions are finite.

1 (a) Let $K \subseteq L$ be a field extension. What does it mean for $\varphi\colon L \to L$ to be a K-automorphism ? **(3 marks)**

(b) (i) For a field extension $K \subseteq L$, define the *Galois group* $\mathrm{Gal}(L/K)$.

(ii) Define when a field extension $K \subseteq L$ is a *Galois extension.* **(4 marks)**

(c) Write down, justifying your answers,

(i) $|\,\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})|$ where $\sqrt[3]{2}$ is the real cube root of 2.

(ii) $|\,\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})|$ where $\omega = \exp(2\pi i/3)$. **(6 marks)**

(d) What is the splitting field of the polynomial $x^3 - 2$ over $\mathbb{Q}$? **(3 marks)**

(e) Justifying your answer carefully,

(i) find a real number γ such that $\mathbb{Q}(\sqrt{3}, \sqrt{7}) = \mathbb{Q}(\gamma)$;

(ii) find the minimal polynomial in $\mathbb{Q}[x]$ of γ. **(7 marks)**

(f) Define what it means for a group G to be *soluble.* **(2 marks)**

2 (a) Define the nth cyclotomic polynomial $\lambda_n(x) \in \mathbb{C}[x]$.

In the remainder of this question you may use without proof the result that

$$x^n - 1 = \prod_{d|n} \lambda_d(x). \tag{1}$$

(b) Explain why any primitive 15th-root of unity will be a root of $x^{10} + x^5 + 1$.

(c) Verify the polynomial identity

$$(x^2 + x + 1)(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1) = x^{10} + x^5 + 1. \tag{2}$$

(d) Explain why it now follows that

$$\lambda_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1.$$

(12 marks)

3 (a) State, without proof, the Theorem of the Primitive Element (TPE). **(2 marks)**

(b) State, without proof, the Theorem that 'Automorphisms Preserve Roots' (APR). **(4 marks)**

(c) Let $K \subseteq L$ be an extension of fields. Using the TPE, or otherwise, prove that if L/K is Galois, then L is the splitting field of a polynomial $f(x) \in K[x]$. **(4 marks)**

In the remainder of this question you may assume without proof the converse of (c): if L is the splitting field of a polynomial $f(x) \in K[x]$, then L/K is Galois.

(d) Let $K \subseteq M \subseteq L$ be field extensions. Prove that if L/K is Galois, then L/M is Galois. **(4 marks)**

(e) Let $K \subseteq M \subseteq L$ be field extensions. Prove that if M/K is Galois, then $\varphi(M) = M$ for all $\varphi \in \mathrm{Gal}(L/K)$. **(5 marks)**

(f) Now assume that in $K \subseteq M \subseteq L$ both L/K and M/K are Galois. Prove that $\mathrm{Gal}(L/M)$ is a normal subgroup of $\mathrm{Gal}(L/K)$ and that the quotient group is isomorphic to $\mathrm{Gal}(M/K)$. State clearly any general result from field theory or group theory which you use. **(19 marks)**

4 Let $\xi = \exp(2\pi i/7)$ and write $\beta = \xi + \xi^{-1}$ and $\gamma = \xi + \xi^2 + \xi^4$.

(a) Find a cubic polynomial with integer coefficients which has β as a root. **(5 marks)**

(b) Express γ^2 in terms of powers of ξ and hence find a quadratic polynomial with integer coefficients which has γ as a root. **(3 marks)**

(c) Show that $\mathbb{Q}(\sqrt{-7}) \subseteq \mathbb{Q}(\xi)$. **(2 marks)**

(d) You are given that $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ has six elements. List the elements, without proof. **(3 marks)**

(e) Choose $\theta \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ such that $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) = \{\mathrm{id}, \theta, \theta^2, \theta^3, \theta^4, \theta^5\}$. Deduce that $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ is cyclic.

Draw the lattice of subgroups for $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$, describing the subgroups in terms of their elements. **(9 marks)**

(f) Draw the lattice of subfields of $\mathbb{Q}(\xi)$ which correspond, by the Fundamental Theorem of Galois Theory, to the lattice of subgroups in (e). **(3 marks)**

End of Question Paper