The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **June 2016**

Probability and Probability Distributions **2 hours**

RESTRICTED OPEN BOOK EXAMINATION.
Candidates may bring to the examination lecture notes and associated lecture material (including set textbooks) plus a calculator that conforms to University regulations.
Candidates should attempt ***ALL*** *questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 80.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Let X be a continuous random variable with distribution function given by

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \ln x & 1 \leq x \leq \mathrm{e} \\ 1 & x > \mathrm{e}. \end{cases}$$

(a) Give the values of $P(X \leq 2)$, $P(X > 4)$ and $P(X = 2)$. ***(3 marks)***

(b) Give the value of $P\left(X \leq \frac{3}{2} \middle| X \leq 2\right)$. ***(2 marks)***

(c) Find the probability density function of X. ***(4 marks)***

(d) Let $Y = X^2$. Find the probability density function of Y. ***(5 marks)***

2 Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be a sample space for an experiment, and assume that each element of S is equally likely to occur.

(a) Define the events $A_1 = \{1, 3, 5, 7\}$, $A_2 = \{1, 2, 3, 4\}$ and $A_3 = \{6, 7, 8\}$.

(i) Give the probabilities of each of the events A_1, A_2 and A_3. ***(3 marks)***

(ii) Which of the following are true? Give reasons for your answers.

(α) A_1 and A_2 are independent.

(β) A_1 and A_3 are independent.

(γ) A_2 and A_3 are independent. ***(6 marks)***

(b) Define a random variable X by saying that if the observed outcome of the experiment is s then the value of X will be $(s - 4)^2$.

(i) Tabulate the probability function of the random variable X. ***(5 marks)***

(ii) Give the mean of the random variable X. ***(3 marks)***

3 Let T be the region defined by $T = \{(x, y) : 0 \leq x \leq y \leq 1$, and let X and Y be random variables with joint probability density function given by

$$f_{X,Y}(x, y) = \begin{cases} 8xy & (x, y) \in T \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(Y \geq 2X)$. ***(5 marks)***

(b) Find the marginal probability density functions of X and Y, and hence find the means of X and Y. ***(10 marks)***

(c) Find the covariance of X and Y. ***(5 marks)***

4 A group of n patients are to be tested for whether or not they have a disease. For each individual patient, the doctor believes that the probability of a positive test is p, which is the same for each patient.

(a) Explain why the number of positive tests might be assumed to have a Binomial distribution with parameters n and p. ***(3 marks)***

(b) If $n = 6$ and $p = 0.3$, give the probability that exactly one of the patients tests positive. ***(2 marks)***

(c) If $n = 1000$ and $p = 0.3$, explain carefully how to use a Normal approximation to find the approximate probability that at least 320 patients test positive. You may give your answer in terms of the standard normal distribution function Φ. ***(5 marks)***

(d) If $n = 1000$ and $p = 0.004$, explain carefully how to use a Poisson approximation to find the approximate probability that exactly 3 patients test positive. ***(5 marks)***

5 A radioactive source is monitored for an hour, and the number of detected emissions from it, x, is counted. The source is known to be one of two substances A or B; if the source is of substance A then theory says that the number of detected emissions should have a Poisson distribution with parameter 3.1, and if the source is of substance B then theory says that the number of detected emissions should have a Poisson distribution with parameter 4.9. The prior probability that the substance is substance A is p.

(a) Give the conditional probability that x emissions are detected given that the source is of substance A. ***(2 marks)***

(b) Give the unconditional probability that x emissions are detected. ***(3 marks)***

(c) Show that the posterior probability that the source is substance A is

$$\frac{1}{1+\frac{1-p}{p}e^{-1.8}\left(\frac{4.9}{3.1}\right)^{x}}.$$

(4 marks)

(d) Under what condition on p will the posterior probability of substance B be higher than that of substance A for all possible observations? ***(5 marks)***

End of Question Paper