MAS6052

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2015–2016**

Stochastic Processes and Finance — **3 hours**

Candidates may bring to the examination a calculator that conforms to University regulations. Full marks may be obtained by complete answers to five questions. All answers will be marked, but credit will be given only for the five best answers.

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1 (a) Give the definition of a measurable function $X : \Omega_1 \to \Omega_2$ between two measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$. ***(2 marks)***

(b) Let $\Omega = \{1,2,3,4,5\}$ with $\mathcal{F} = \mathcal{P}(\Omega)$, the power set of Ω. The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is the uniform probability measure: $\mathbb{P}(\{i\}) = 0.2$ for all $i = 1, 2, \cdots, 5$. Let

$$A = \{1,2,3\}, \quad B = \{3,4,5\}.$$

(i) What is the smallest sub σ-algebra of $\mathcal{F}$ containing A and B? ***(4 marks)***

(ii) Compute $\mathbb{E}\big[2\mathbf{1}_A + 3\mathbf{1}_B\big]$ and $\mathbb{E}[\mathbf{1}_A \cdot \mathbf{1}_B]$. ***(4 marks)***

(c) Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$ such that $\mathbb{P}(X \leq c) \in \{0,1\}$ for every $c \in \mathbb{R}$.

(i) Show that there exists $\alpha \in \mathbb{R}$ such that $\mathbb{P}(X = \alpha) = 1$.
Hint: Look at the cumulative distribution function of X. ***(6 marks)***

(ii) What is $\mathbb{E}X$? ***(1 mark)***

(iii) Is the probability distribution p_X on $\mathbb{R}$ induced by the random variable X absolutely continuous with respect to the standard normal probability distribution? Explain. ***(3 marks)***

2 (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a square integrable random variable (i.e. $\mathbb{E}(X^2) < \infty$) defined on it. Let Y be another random variable which is measurable with respect to a sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x)| \leq 10$ for all $x \in \mathbb{R}$. Show that

$$\mathbb{E}\left[\left(X - f(Y)\right)^2\right] = \mathbb{E}\left[\left(X - \mathbb{E}(X|\mathcal{G})\right)^2\right] + \mathbb{E}\left[\left(\mathbb{E}(X|\mathcal{G}) - f(Y)\right)^2\right]$$

Hint: Write $X - f(Y) = \{X - \mathbb{E}(X|\mathcal{G})\} + \{\mathbb{E}(X|\mathcal{G}) - f(Y)\}$. ***(5 marks)***

(b) Let $X_1, X_2, \cdots$ be i.i.d. random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 0.5$. Let $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$. Show that $S_n^2 - n$ is a martingale with respect to the filtration $(\mathcal{F}_n,\ n \geq 0)$ where $\mathcal{F}_n = \sigma\{X_1, X_2, \cdots, X_n\}$, $n \geq 1$ and $\mathcal{F}_0$ is the trivial σ-algebra. ***(5 marks)***

(c) Let $(B(t),\ t \geq 0)$ denote a Brownian motion. Show that $X(t) = \exp\left[B(t) - (t/2)\right]$ is a martingale with respect to the natural filtration.
Note: For $Z \sim N(0, \sigma^2)$ *we have* $\mathbb{E}(e^Z) = \exp(\sigma^2/2)$. ***(5 marks)***

(d) Consider a filtration $(\mathcal{F}_n, n \in \mathbb{Z}_+)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) Define what it means to be a (discrete) stopping time with respect to the filtration. ***(2 marks)***

(ii) Show that if S, T are (discrete) stopping times with respect to the filtration, then so is $S + T$. ***(3 marks)***

3 Consider a *Binomial Asset Pricing Model* with two assets, a bond (or bank account) and a stock. The rate of interest of the bank/bond is 10% (i.e. $r = 0.1$). The initial stock price is £100 and the possible rates of interest of the stock are −50% (i.e. $d = -0.5$) and 100% (i.e. $u = 1$), both of which are equally likely. We are interested in pricing a **European call option** with maturity time 2 and strike price £200, that is $X = \left(S(2) - 200\right)_+ = \max\left(S(2) - 200, 0\right)$.

(a) Find the arbitrage-free price process for the option. ***(10 marks)***

(b) Find a replicating strategy for the option. ***(10 marks)***

4 (a) Consider a *Binomial Asset Pricing Model* with two assets, a bond (or bank account) and a stock. The rate of interest of the bank/bond is 10% (i.e. $r = 0.1$). The initial stock price is £100 and the possible rates of interest of the stock are -50% (i.e. $d = -0.5$) and 100% (i.e. $u = 1$), both of which are equally likely.

Compute the arbitrage-free price for buying an **American put option** (at time 0) with terminal time 2 and strike price £200. Note that the payoff of this option at time n is $X(n) = \big(200 - S(n)\big)_+ = \max\big(200 - S(n), 0\big)$. ***(10 marks)***

(b) Consider a finite market model with terminal time T, which is arbitrage-free. Consider *European call* and *European put* options with strike price k for a certain stock. Denote by S_n the price of the stock at time n, $0 \leq n \leq T$. Let C_n be the arbitrage-free price of the call option and P_n be the arbitrage-free price of the put option at time n. Let r be the rate of interest for the bank so that £1 put in the bank now will give £$(1+r)^n$ at time n. Show that we must have the following equation called the *put-call parity*

$$C_n - P_n = S_n - k(1+r)^{-(T-n)} \quad \text{for all } n.$$

Hint: One possible approach is by creating an arbitrage opportunity if the above relation does not hold. You can buy and sell options. ***(5 marks)***

(c) Consider a (discrete-time) finite market with terminal time T. There are $d+1$ assets with prices given by the process $S(n) = \big(S_0(n), S_1(n), \cdots, S_d(n)\big)$ where $n = 0, 1, \cdots T$. Here S_0 is the price of a risk-free asset and $S_1, S_2, \cdots, S_d$ are the prices of stocks. Let $\phi(n) = \big(\phi_0(n), \phi_1(n), \cdots, \phi_d(n)\big)$ be a trading portfolio.

(i) What is the *value of the portfolio* or the *wealth process*? ***(2 marks)***

(ii) What equation is satisfied by a self-financing portfolio? ***(3 marks)***

5 Let $(B(t),\, t \geq 0)$ denote a Brownian motion.

(a) Consider the process $F(t)$, $0 \leq t \leq 3$ given by

$$F(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 2 & \text{if } 1 < t \leq 2, \\ 3 & \text{if } 2 < t \leq 3, \end{cases}$$

and let $I_3(F) = \int_0^3 F(s)\, dB(s)$.

(i) Express $I_3(F)$ in terms of $B(1), B(2)$ and $B(3)$. ***(2 marks)***

(ii) Compute the mean and variance of $I_3(F)$. ***(2 marks)***

(iii) Show that $I_3(F)$ is normally distributed. ***(2 marks)***

(iv) Compute $\mathbb{E}[I_3(F) \big| \mathcal{F}_{\frac{3}{2}}]$ where $(\mathcal{F}_t,\, t \geq 0)$ is the natural filtration of Brownian motion. ***(2 marks)***

(b) (i) Find the stochastic differential satisfied by $X(t) = \sqrt{1 + B(t)^2}$. ***(4 marks)***

(ii) Use Itô's product formula to find the stochastic differential of $B(t)X(t)$. ***(4 marks)***

(c) Find the solution to the SDE

$$dY(t) = -4Y(t)\, dt + 5\, dB(t), \quad Y(0) = 3.$$

(4 marks)

6 Let $(B(t), t \geq 0)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a Black-Scholes model where the price of the bond at time t is $A(t) = e^{2t}$ and the price of the stock at time t is given by the SDE: $dS(t) = S(t)dt + S(t)dB(t)$ with $S(0) = 1$. A portfolio is an adapted process $\big(\phi(t), \psi(t)\big)$ where $\phi(t)$ is the number of stocks held at time t and $\psi(t)$ is the number of bonds held at time t.

(a) Write down the equation satisfied by the wealth process in a self-financing portfolio. ***(2 marks)***

(b) Show that a portfolio with associated wealth process $V(t)$ is self-financing if and only if $d\tilde{V}(t) = \phi(t)\, d\tilde{S}(t)$. Here $\tilde{S}(t) = e^{-2t}S(t)$ is the discounted stock price and $\tilde{V}(t) = e^{-2t}V(t)$ is the discounted wealth process. ***(6 marks)***

(c) Define the measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(B(t) - \frac{t}{2}\right).$$

By Girsanov's theorem we know that the process $W(t) = B(t) - t$ is a martingale under the measure $\mathbb{Q}$.

(i) Find the SDE satisfied by $\tilde{S}$ and show that $\tilde{S}$ is a martingale under the measure $\mathbb{Q}$. ***(5 marks)***

(ii) Give an explicit expression for $\tilde{S}(t)$ under the measure $\mathbb{Q}$. ***(2 marks)***

(iii) Consider now a European call option with terminal time 2 and strike price 1, that is $X = \big(S(2) - 1\big)_+$. Find the arbitrage-free price of the option at time 0. (*It is enough to give your answer as an integral*) ***(5 marks)***

End of Question Paper