



The
University
Of
Sheffield.

MAS6352

SCHOOL OF MATHEMATICS AND STATISTICS

**Spring Semester
2015–2016**

MAS6352 Analysis II

2 hours

Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.

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1 (i) Let S be a given set and consider the following statements:

(I) A σ -algebra is a collection Σ of subsets of S for which $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, whenever $A_n \in \Sigma$ for all $n \in \mathbb{N}$.

(II) A measure is a mapping $m : \Sigma \rightarrow [0, \infty)$ for which

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n),$$

whenever (A_n) is a sequence of subsets of S .

These statements are both wrong. Explain carefully how they can be corrected. (7 marks)

(ii) Let Σ_1 and Σ_2 be σ -algebras of a set S .

(a) Define

$$\Sigma_1 \cap \Sigma_2 = \{A \subseteq S, A \in \Sigma_1 \text{ and } A \in \Sigma_2\}.$$

Show that $\Sigma_1 \cap \Sigma_2$ is a σ -algebra. (3 marks)

(b) Define

$$\Sigma_1 \cup \Sigma_2 = \{A \subseteq S, A \in \Sigma_1 \text{ or } A \in \Sigma_2\},$$

(where “or” is inclusive). Either show that $\Sigma_1 \cup \Sigma_2$ is a σ -algebra, or give a counter-example to demonstrate that, in general, it isn't.

(3 marks)

(iii) Recall that a set is *countable* if it can be put into one-to-one correspondence with the natural numbers. Let S be a set and Σ be a collection of subsets of S that is chosen as follows: $A \in \Sigma$ if either A is finite or countable or A^c is finite or countable. Show that Σ is a σ -algebra. You may use the facts that a countable union of finite or countable sets is itself finite or countable, and that a subset of a countable set is finite or countable. (8 marks)

(iv) Define the *symmetric difference* $A \Delta B$ between subsets A and B of S by

$$A \Delta B = (A \cup B) - (A \cap B).$$

(a) Show that $A \Delta B = (A - B) \cup (B - A)$. (5 marks)

(b) Calculate the Lebesgue measure of $A \Delta B$ when $A = (0, 1)$ and $B = (-1/3, 1/2) \cup (3/4, 2)$. (4 marks)

(c) If A and B are as in (b), and $S = (-1, 2)$ equipped with the uniform probability measure on its Borel σ -algebra, what is the probability of the set $A \Delta B$? (3 marks)

2 Throughout this question (S, Σ, m) is a measure space and \mathbb{R} is equipped with its usual Borel σ -algebra. Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is denoted by λ .

(i) Let $f : S \rightarrow \mathbb{R}$ be measurable. Explain how the Lebesgue integral of f is constructed in each of the following cases, carefully stating any restrictions on f that are needed (if necessary), and giving the range of values that the integral may take:

(a) f is a non-negative simple function, **(3 marks)**

(b) f is a non-negative measurable function, **(2 marks)**

(c) f is a general measurable function. **(3 marks)**

What does it mean for f to be *integrable*? **(1 mark)**

(ii) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as follows

$$f = \begin{cases} 0 & \text{if } x < -2 \\ -7, & \text{if } -2 \leq x < -1, \\ 4 & \text{if } -1 \leq x < 0, \\ 11 & \text{if } 0 \leq x < 1, \\ -3 & \text{if } 1 \leq x < 2, \\ -2 & \text{if } 2 \leq x < 5, \\ 0, & \text{if } x \geq 5 \end{cases}$$

Write $|f|$ as a simple function and hence calculate $\int_S |f| d\lambda$. **(3 marks)**

(iii) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is measurable and $g(x) = \frac{xf(x)}{1+x^2}$ for all $x \in \mathbb{R}$, explain why g is integrable. **(5 marks)**

(iv) Suppose that $g : S \rightarrow (0, \infty)$ is a measurable function.

(a) Prove that $1/g$ is measurable, where $(1/g)(x) = 1/g(x)$, for all $x \in S$. **(6 marks)**

(b) If $f : S \rightarrow \mathbb{R}$ is measurable, show that $h = f/g$ is measurable. **(3 marks)**

(v) Let (f_n) be a sequence of measurable functions from S to \mathbb{R} and define

$$A = \{x \in S; \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}.$$

Show that $A \in \Sigma$. [Hint: Make use of the measurable functions $\liminf_{n \rightarrow \infty} f_n$ and $\limsup_{n \rightarrow \infty} f_n$.] **(7 marks)**

- 3** (i) Let (S_1, Σ_1, m_1) and (S_2, Σ_2, m_2) be measure spaces. A version of *Fubini's theorem* is as follows: Let $f : S_1 \times S_2 \rightarrow \mathbb{R}$ be a non-negative measurable function. Then the mappings

$$x \rightarrow \int_{S_2} f(x, y) dm_2(y) \quad \text{and} \quad y \rightarrow \int_{S_1} f(x, y) dm_1(x),$$

are both measurable. Furthermore

$$\begin{aligned} \int_{S_1 \times S_2} f d(m_1 \times m_2) &= \int_{S_1} \left(\int_{S_2} f(x, y) dm_2(y) \right) dm_1(x) \\ &= \int_{S_2} \left(\int_{S_1} f(x, y) dm_1(x) \right) dm_2(y). \end{aligned}$$

- (a) Explain briefly why this theorem is true in the case where f is an indicator function. You should not give a detailed proof, but your explanation should include definitions of key concepts such as x and y -slices of a set (where $x \in S_1$ and $y \in S_2$), and product measure. **(9 marks)**
- (b) Using the result of (a), give a detailed proof of the theorem for general non-negative measurable functions f . **(8 marks)**
- (c) Prove Fubini's theorem for a real-valued, integrable function $f : S_1 \times S_2 \rightarrow \mathbb{R}$. **(6 marks)**
- (ii) (a) Explain why $(x, y) \rightarrow e^{-x-y} \frac{xy^2}{(1+x^2)(1+y^2)}$ is integrable (with respect to Lebesgue measure) on $(0, 1) \times (0, 1)$. Do not attempt to evaluate the integral. **(4 marks)**
- (b) Evaluate $\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 e^{-\left(\frac{x+y}{n}\right)} \frac{nxy^2}{(1+x^2)(n+y^2)} dx dy$. **(6 marks)**

4 Throughout this question, (Ω, \mathcal{F}, P) is a probability space.

- (i) Let (A_n) be a sequence of subsets of Ω in \mathcal{F} .
- (a) Define the sets $\limsup_{n \rightarrow \infty} A_n$ and $\liminf_{n \rightarrow \infty} A_n$, and briefly explain why each set is in \mathcal{F} . (3 marks)

- (b) Give a direct proof that $\limsup_{n \rightarrow \infty} P(A_n) \leq P\left(\limsup_{n \rightarrow \infty} A_n\right)$. (5 marks)

- (c) If $B \in \mathcal{F}$, show that

$$B - \liminf_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} (B - A_n).$$

Hence deduce that $\left(\liminf_{n \rightarrow \infty} A_n\right)^c = \limsup_{n \rightarrow \infty} A_n^c$. (7 marks)

- (ii) If $X : \Omega \rightarrow \mathbb{R}$ is an integrable random variable and $a \in \mathbb{R}$, show that

$$\mathbb{E}(\min\{X, a\}) \leq \min\{\mathbb{E}(X), a\}.$$

(6 marks)

Use this inequality to find an upper bound for $\mathbb{E}(\min\{X, a\})$ when

- (a) $a = 1$, and X is a Bernoulli random variable taking values 1 with probability $3/4$ and 0 with probability $1/4$, (1 mark)
- (b) $a = 54$, and $X = Y_1 + Y_2 + \dots + Y_{10}$ with $Y_k \sim N(k, 1)$ for $k = 1, 2, \dots, 10$. (3 marks)
- (iii) State *Lebesgue's dominated convergence theorem* in a probabilistic context. (3 marks)

- (iv) A sequence (X_n) of random variables is said to *converge in the mean* to a random variable X if $\lim_{n \rightarrow \infty} \mathbb{E}(|X_n - X|) = 0$. Let $(A_n, n \in \mathbb{N})$ be a sequence of subsets of Ω for which $A_n \in \mathcal{F}$, $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} A_n = \Omega$.

- (a) Show that $(\mathbf{1}_{A_n}, n \in \mathbb{N})$ converges pointwise to 1 on Ω . (2 marks)
- (b) Let X be an integrable random variable and define $X_n = X\mathbf{1}_{A_n}$ for each $n \in \mathbb{N}$. Prove that $(X_n, n \in \mathbb{N})$ converges in mean to X . (3 marks)

End of Question Paper