The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Autumn Semester 2015–16**

Mathematics Core 1 — **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f((a, b)) = a^2 + b^2$.

(i) Is f injective? Justify your answer. ***(1 mark)***

(ii) Let $S = \{(a, b) \in \mathbb{N} \times \mathbb{N} : \ a + b = 4\}$. What are the cardinalities of S and $f(S)$? ***(2 marks)***

2 Prove by using induction that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

for all $n \in \mathbb{N}$. ***(5 marks)***

3 Write down the expression for $f'(x)$, the derivative of a function f at x, as a suitable limit. Using this, show that if $f(x) := \dfrac{1}{1-2x}$ then $f'(x) = \dfrac{2}{(1-2x)^2}$. ***(4 marks)***

4 (i) Determine $\lim\limits_{x \to 0^+} x\left(1 + \cos\left(\dfrac{1}{x}\right)\right)$ using the sandwich rule. ***(2 marks)***

(ii) Determine $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$ using L'Hôpital's rule. ***(2 marks)***

5 Show that

$$\int \frac{1}{\sqrt{x(x+1)}}dx = 2\ln(\sqrt{x}+\sqrt{x+1}) + C$$

where C is a constant. (**Hint:** Differentiate!) Hence, or otherwise, find

$$\int \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x+1}}\right)^2 dx.$$

(5 marks)

6 (i) Let $z_1 = 1+2i, z_2 = 2+3i, z_3 = -1+3i$ and $z_4 = 3-4i$. Express each of the following in the form $a+bi$, where $a, b \in \mathbb{R}$:

$$z_1 + \overline{z_2}; \quad (z_1+\overline{z_2})z_3; \quad \frac{(z_1+\overline{z_2})z_3}{z_4}.$$

(3 marks)

(ii) Indicate the regions $\{z \in \mathbb{C} : z + \overline{z} = 0\}$ and $\{z \in \mathbb{C} : |z| = 1\}$ on an Argand diagram. ***(2 marks)***

(iii) Show that if $z \neq \pm 1$ has modulus 1 then $\dfrac{z}{1-z^2}$ is purely imaginary.

(3 marks)

7 Write down the Maclaurin series expansion for $\dfrac{1}{1+x^2}$ and use it to show that

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots.$$

Hence or otherwise derive the Maclaurin series expansion for

$$x\tan^{-1} x - \frac{1}{2}\ln(1+x^2).$$

You do not need to worry about the radii of convergence of the series involved.

(6 marks)

8 State Euler's identity for $e^{i\theta}$. By raising it to the n-th power deduce de Moivre's Theorem and use it to show that

$$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

Hence show that

$$\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}.$$

(9 marks)

9 In this question y is a function of t. Thus $y' = \dfrac{dy}{dt}$ and $y'' = \dfrac{d^2y}{dt^2}$. You should assume that the variable is always positive i.e. $t > 0$.

(i) Find the general solution to the differential equation $y'' - 6y' + 9y = 0$. ***(2 marks)***

(ii) Find a particular solution to the differential equation $y'' - 6y' + 9y = e^{3t}$. ***(3 marks)***

(iii) By using a substitution of the form $y^2 = tu$, find a solution to the differential equation $2tyy' = y^2 + 2t^3 + t$ which satisfies $y = 1$ when $t = 1$. ***(5 marks)***

10 Let A and B be non-empty finite sets. Write $|A| = m$ and $|B| = n$.

(i) How many different functions from A to B are there? ***(1 mark)***

(ii) How many different injective functions are there from A to B if $m \leq n$? What will your answer be if $m > n$? ***(2 marks)***

(iii) How many different surjective functions from A to B are there if $n = 2$ and $m \geq 2$? ***(3 marks)***

You do not have to justify your answers to the first two parts.

End of Question Paper