Data provided: formula sheet

The University Of Sheffield.

MAS241

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

Mathematics II (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) The electrical charge in a circuit is described by

$$y''(t) + y(t) = H(t-1)$$

subject to the initial conditions $y(0) = y'(0) = 0$.

(a) Show that the Laplace transform of $y(t)$ is given by

$$Y(s) = \frac{e^{-s}}{s(s^2+1)}.$$

(10 marks)

(b) Use part (a) to find $y(t)$. **(6 marks)**

(ii) Let $f(t) = \frac{1}{2}H(t+1)H(1-t)$. Find $\mathcal{F}\{f * f(t)\}$. **(4 marks)**

2 (i) Consider the periodic function $f(t)$ with fundamental period 2π defined by

$$f(t) := \begin{cases} 0 & \text{if } -\pi < t \leq 0, \\ t & \text{if } 0 < t \leq \pi. \end{cases}$$

Find the Fourier series of $f(t)$ and sketch its graph over the interval $[-2\pi, 3\pi]$. Your sketch should indicate the values of the series at points of discontinuity. **(16 marks)**

(ii) The exponential form of the Fourier series of a function $g(t)$ with period T is given by

$$S[g](x) := \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n x}$$

where

$$c_n := \frac{1}{T}\int_{-T/2}^{T/2} g(x)e^{-j\omega_n x}dx \quad \text{and} \quad \omega_n = \frac{2\pi n}{T}.$$

The Fourier series of the 2π periodic function defined by $f(x) = x - x^2$ on $[-\pi, \pi]$ is given by

$$S[f](x) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^{n+1}}{n^2}\cos(nx) + \frac{2(-1)^{n+1}}{n}\sin(nx)\right).$$

Find the exponential form of the Fourier series for $f(x)$. **(4 marks)**

3 (i) Let $f(x,y) = (x^2+6xy+9y^2)^2$. By calculating both $f_{xy}(x,y)$ and $f_{yx}(x,y)$ directly, verify that $f_{xy}(x,y) = f_{yx}(x,y)$. **(6 marks)**

(ii) Find and classify all the critical points of the function

$$f(x,y) = e^{x-xy+y}.$$

(10 marks)

(iii) Let $f(x,y)$ be a function of two variables, and let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Show that

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta).$$

Hence or otherwise, find an expression for $\frac{\partial^2 f}{\partial r^2}$ in terms of r and θ when $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x^2} = 1$. **(4 marks)**

4 (i) Let $T \subset \mathbb{R}^2$ be the region bounded by the lines $y = x$, $x = 0$, and $y = 1$, and let $f(x,y) = xy$. Find

$$\iint_T f(x,y)dA.$$

(10 marks)

(ii) Use integration to find the area of the surface S described by the equation $z = \sqrt{1-x^2-y^2}$ above the region $D = \{(x,y) \,|\, x^2+y^2 \leq 1\}$. **(10 marks)**

5 (i) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} = (xyz, xy+yz+xz, x+y+z).$$

(a) Calculate **div F**. **(5 marks)**

(b) Calculate **curl F**. **(5 marks)**

(c) Calculate **div curl F**. **(5 marks)**

(ii) Let

$$f(x,y) = 2x^2 - 3y.$$

(a) Calculate the directional derivative of $f(x,y)$ at $(1,1)$ in the $\mathbf{v} = (a,b)$ direction. **(3 marks)**

(b) For which unit vector $\mathbf{v} = (a,b)$ is the directional derivative of $f(x,y)$ maximised at $(1,1)$? **(2 marks)**

End of Question Paper

MAS241 FORMULA SHEET

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^{\infty} e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$ (for $a > 0$)	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$ (for $a > 0$)	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
t^n (for $n \geq 0$)	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t-T)$ (for $T \geq 0$)	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t-T)$ (for $T \geq 0$)	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega - \theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a \rvert} F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t \rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(\omega)$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r, \theta, z) = \Big(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z \Big)$$
$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho, \theta, \phi) = \Big(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho}) \Big)$$
$$dV = \rho^2 \sin(\phi) d\rho d\phi d\theta.$$