Data provided: Formula sheet

MAS248

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.
The paper is marked out of a total of 60 marks.

1 (i) The temperature $T(x, y, z)$ of a gas in a reactor is given by the equation

$$T = x^2 - y^2 + xyz + 273.$$

Find the direction in which the temperature increases most rapidly at the point $(1, -2, 3)$.
What is the rate of the increase in temperature in this direction?
(2 marks)

(ii) Find the directional derivative of the scalar field $q(x, y, z)$ which is given by

$$q = x^2 z + xy$$

at the point $(-1, 2, 1)$ in the direction of the vector $\mathbf{a} = (2, -2, 1)$.
(3 marks)

(iii) A force field $\mathbf{F}(x, y, z)$ is given by

$$\mathbf{F} = (3x^2yz - 3y, x^3z - 3x, x^3y + 2z).$$

(a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$. ***(2 marks)***

(b) Find a scalar potential ϕ such that $\mathbf{F} = \nabla\phi$. ***(5 marks)***

(iv) (a) Compute the divergence of the vector field $\mathbf{h}(x, y, z) = (x^3, y^3, z^3)$.
(1 mark)

(b) The scalar field $\psi(x, y)$ is given by

$$\psi = x^3 - 3xy^2 + y^3.$$

Calculate $\nabla^2\psi$. ***(2 marks)***

2 (i) Find and classify the stationary points of the function

$$f(x,y) = 4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4.$$

(9 marks)

(ii) The function $g(x,y)$ satisfies the differential equation

$$y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = 0.$$

By changing to new variables, $u = x^2 - y^2$ and $v = 2xy$, show that g is, in fact, a function of $x^2 - y^2$ only. ***(6 marks)***

3 (i) Write down the iteration formula for the Newton-Raphson method. Starting with an initial guess of $x_0 = 0.5$, use the Newton-Raphson method to solve the equation

$$\cos x = 2x,$$

correct to four decimal places. ***(4 marks)***

(ii) A periodic function, $f(t)$, with period 2π is defined by

$$f(t) = t \quad \text{for} \quad 0 \le t < 2\pi, \quad f(t) = f(t + 2\pi).$$

Sketch a graph of the function $f(t)$ for values of t from $t = -4\pi$ to $t = 4\pi$. ***(2 marks)***

Obtain the Fourier series expansion for $f(t)$. ***(9 marks)***

4 (i) Show that the partial differential equation

$$6\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 u}{\partial t^2} = 0$$

has solutions of the form $u(x,t) = f(x+2t) + g(x-3t)$, where f and g are arbitrary twice differentiable functions. ***(4 marks)***

Given that for all x, u satisfies the conditions

$$u(x,0) = x^2 - 1$$

and

$$\frac{\partial u}{\partial t}(x,0) = 2x,$$

find the solution for u. ***(11 marks)***

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi = \text{grad } \phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x,y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x,y)$, where $x = x(u,v)$, $y = y(u,v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u,v)$, where $u = u(x,y)$, $v = v(x,y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x,y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x,y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$