Data provided: Formula sheet

MAS252

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

Further Civil Engineering Mathematics and Computing **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Use the properties of trigonometric functions to verify the identities

(i)

$$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & \text{if} \quad m \neq n \\ 1, & \text{if} \quad m = n \neq 0 \\ 2, & \text{if} \quad m = n = 0 \end{cases},$$

(8 marks)

(ii)

$$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & \text{if} \quad m \neq n \\ 1, & \text{if} \quad m = n \neq 0 \\ 0, & \text{if} \quad m = n = 0 \end{cases},$$

(7 marks)

(iii)

$$\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0.$$

(2 marks)

(iv) Hence prove the identity that for any integrable function $f(x)$ we have

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where a_0, a_n and b_n are the coefficients in a Fourier series expansion of the function $f(x)$. ***(8 marks)***

2 Use the method of separation of variables to find the solution of the heat conduction equation

$$100\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad u = u(x,t), \quad 0 < x < 1,$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = \sin 2\pi x - \sin 5\pi x, \quad 0 \le x \le 1$$

(25 marks)

3 (i) Determine the first three non-zero terms of the series solution around $x = 1$ of the differential equation

$$x^2 y'' + (1+x)y' + 3y\ln x = 0$$

subject to the conditions $y(1) = 2$ and $y'(1) = 0$ ***(16 marks)***

(ii) Perform *seven* iterations of the bisection method to find an approximate root of the function

$$f(x) = e^{-x}(3.2\sin x - 0.5\cos x)$$

in the interval $[3, 4]$. Give your answer correct to *four* decimal places.

(9 marks)

4 (i) A cylindrical tank has a 1 m height (h) and a radius of 0.3 m (r). If the height is increased by 5 cm and the radius by 1 cm, use the small error formula to calculate the change in the volume of the cylinder. ***(6 marks)***

(ii) Show that the following functions satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(a) $u(x,y) = e^{ax}\sin ay$, (b) $u(x,y) = \ln(x^2 + y^2)$

(11 marks)

(iii) If $V = V(x,y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}$$

(8 marks)

End of Question Paper

Formula sheet

- The volume of a cylinder of height h and radius r is given by

$$V = \pi r^2 h$$

- $$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

- $$\frac{d}{dx}\tan^{-1} u(x) = \frac{u'(x)}{1+u^2(x)}$$

- The local truncation error in the case of the 4th order Runge-Kutta method is given by

$$Y(x) - y(x) = Ch^4$$

where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \le x \le l$, then the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

If the function $f(x)$ is defined over the interval $0 \le x \le l$, then the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

while the sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$