Data provided: Formula sheet

MAS253

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015-2016**

Mathematics for Engineering Modelling **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Find the sum of the infinite series

$$3 + x^{\frac{1}{2}} + \frac{1}{3}x + \frac{1}{9}x^{\frac{3}{2}} + \frac{1}{27}x^2 + \ldots$$

and determine its radius of convergence. Hence find the sum of the infinite series

$$3x^{-\frac{1}{2}} + 2 + x^{\frac{1}{2}} + \frac{4}{9}x + \ldots$$

(8 marks)

(ii) Use l'Hôpital's rule to evaluate

(a) $$\lim_{\theta \to 0} \frac{1 - \cos(\theta^2)}{\theta^4},$$

(b) $$\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right).$$

(7 marks)

(iii) The Maclaurin series for $\sin x$ is

$$x - \frac{1}{3!}x^3 + \ldots$$

Calculate the Maclaurin series for $\cos x$ including terms up to x^4. Hence show, including terms up to x^4, that

$$\cos x \cdot \sin x = \frac{1}{2}\sin(2x).$$

Find the first 2 terms of the Taylor series for $1/\sin(x)$ about the point $x = \pi/4$. Use the result to find an approximate value for $1/\sin(2\pi/9)$, quoting your result to 3 significant figures. ***(10 marks)***

2 Consider the function

$$f(x) = \begin{cases} 0 & 0 < x < 2 \\ 4 - x & 2 \le x < 4 \end{cases}$$

(i) Sketch the Fourier series $F(x)$ of the function $f(x)$ for the range $-4 < x < 8$.
(4 marks)

(ii) Show that

$$F(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{m=0}^{\infty} \frac{\cos((2m+1)\pi x/2)}{(2m+1)^2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x/2)}{n}\,.$$

(16 marks)

(iii) State Fourier's Theorem (Dirichlet's Theorem for the case of a Fourier series) and deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

(5 marks)

3 (i) Calculate *by integration* the Laplace transform of $f(t) = 3t + e^{2t}$, and determine the condition on s for the result to be valid.
(6 marks)

(ii) With the aid of the Table of Laplace transforms, find (a) the inverse Laplace transform of

$$\frac{s-3}{(s+3)(s-5)}\,,$$

and (b) the Laplace transform of

$$u(t-2)\,\mathrm{e}^{3t}\,,$$

where $u(t)$ is the unit Heaviside function. ***(9 marks)***

(iii) A ruler is clamped at one end to a table, the other end extends over the edge of the table and is free to vibrate. The displacement, $y(t)$, of the tip of the free end of the ruler obeys the governing equation

$$\ddot{y}(t) + 4\,\dot{y}(t) + 68\,y(t) = f(t)\,,$$

where the over-dot denotes the derivative with respect to time, and $f(t)$ is the force applied to the tip. Initially at rest with zero displacement, the tip is then subject to a force of magnitude 102 for 0.5 time units.
Use the method of Laplace transforms to find the displacement $y(t)$.
(10 marks)

4 The position of a vibrating string evolves according to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

The ends of the string at $x = 0$ and $x = a$ are fixed at $u = 0$, and initially the string is at rest.

(i) Using the *method of separation of variables*, show that the general solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x \cos\frac{n\pi}{a}ct, \qquad (1)$$

where the A_n are constants. ***(16 marks)***

(ii) If $a = \pi$ and initially $u = \varepsilon \sin(3x)\cos(x)$, where ε is a constant, determine the A_n for this case. ***(5 marks)***

(iii) Show that the general solution (1) can be expressed in terms of waves travelling to the left and right at speed c. ***(4 marks)***

5 (i) Let $x = r\cos\theta$ and $y = r\sin\theta$. Calculate

$$\left(\frac{\partial y}{\partial r}\right)_\theta \quad \text{and} \quad \left(\frac{\partial y}{\partial r}\right)_x .$$

Hence show that their product is 1. ***(6 marks)***

(ii) Evaluate the integral

$$\int_0^3 \int_{-2}^{4-2x} \frac{4-2x}{y^2}\, \mathrm{d}y\, \mathrm{d}x\, .$$

(6 marks)

(iii) A region R is given by the limits $x^2 + y^2 \geq 4,\ x \leq y \leq 3$ and $x \geq 0$. *Using a change of coordinates*, evaluate the integral

$$\iint_R \frac{x\, y^2}{(x^2+y^2)^{\frac{3}{2}}}\, \mathrm{d}x\, \mathrm{d}y\, .$$

(13 marks)

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2\sin P \cos P,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2\cos^2 P - 1 = 1 - 2\sin^2 P,$$

$$\cos P \cos Q = \frac{1}{2}\{\cos(P+Q) + \cos(P-Q)\},$$

$$\sin P \sin Q = -\frac{1}{2}\{\cos(P+Q) - \cos(P-Q)\},$$

$$\sin P \cos Q = \frac{1}{2}\{\sin(P+Q) + \sin(P-Q)\}.$$

Geometric progression

The partial sum to n terms of

$$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$$

is

$$S_n = a(1-r^n)/(1-r), \quad r \neq 1 \quad .$$

Taylor Series for functions of one variable (x)

The Taylor series of $f(x)$ about $x=a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series of $f(x)$ is the special case of the Taylor series when $a=0$:

$$f(x) = f(0) + f'(0)\, x + \frac{1}{2!} f''(0)\, x^2 + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots \qquad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots. \qquad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots. \qquad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots \qquad (R = 1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots. \qquad (R = 1)$$

R is the radius of convergence.

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots..$$

If n is positive and integer, series terminates.

If n is negative or non-integer (or both), the series is an infinite series with the radius of convergence, $R = 1$.

Fourier Series

The Fourier series of $f(x)$ for $-l < x < l$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad ,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\,, \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\,, \quad n = 1, 2, \ldots$$

Laplace Transform

The Laplace Transform of $f(t)$ is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt \quad .$$

For special cases, see later page.

Partial Differentiation

$$\delta F = F(x+\delta, y+\varepsilon) - F(x,y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that $F = F(x,y)$ and that x and y are functions of t, i.e. $x = x(t), y = y(t)$, then
$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} .$$
2. Suppose that $F = F(x,y)$ and that x and y are functions of the variables u and v, i.e. $x = x(u,v), y = y(u,v)$, then
$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}; \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} .$$

Taylor Series for functions of two variables (x, y)

The Taylor series of $f(x,y)$ about $x = a, y = b$ is

$$\begin{aligned} f(x,y) &= f(a,b) + \{(x-a) f_x(a,b) + (y-b) f_y(a,b)\} + \\ &+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + \\ &+ (y-b)^2 f_{yy}(a,b)\} + \\ &+ \ldots \end{aligned}$$

Here $f_x = \frac{\partial f}{\partial x}$ etc.

Partial Differential Equations (2 independent variables)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t}$$ Heat conduction (or diffusion) eqn.

equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ Wave equation

General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A\cos \omega x + B\sin \omega x$$

$$X'' = \omega^2 X \Rightarrow X(x) = C\cosh \omega x + D\sinh \omega x$$

or $Ee^{\omega x} + Fe^{-\omega x}$

$$T' = kT \Rightarrow T(t) = Ae^{kt}$$

Table of Laplace Transforms	
$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{iv}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!\ (n \geq 1)$	$1/s^n$
e^{at}	$\frac{1}{s-a}$
$\frac{1}{a}\sin at$	$\frac{1}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\frac{1}{a}\sinh at$	$\frac{1}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{\sin at - at\cos at}{2a^3}$	$\frac{1}{(s^2+a^2)^2}$
$\frac{t\sin at}{2a}$	$\frac{s}{(s^2+a^2)^2}$
$e^{at} f(t)$	$F(s-a)$, where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$u(t-a)\ f(t-a)$	$e^{-as} F(s)$, where $F(s) = L(f(t))$