The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2015–2016

Continuum Mechanics — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Give the definition of a *unitary* linear operator in a vector space. ***(1 mark)***

(ii) Give the definition of an *orthogonal* matrix. Show that the linear operator $\mathbf{B}$ is unitary if and only if its matrix $\hat{\mathbf{B}}$ with respect to a particular basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ is orthogonal. When is the operator $\mathbf{B}$ called *proper* unitary? ***(7 marks)***

(iii) (a) Give the definition of a *positive-definite* linear operator. ***(1 mark)***

(b) State the polar decomposition theorem. ***(3 marks)***

(iv) You are given that the polar decomposition of the operator $\mathbf{T}$ is given by $\mathbf{T} = \mathbf{BU}$, where the matrices of the operators $\mathbf{B}$ and $\mathbf{T}$ are defined by

$$\hat{\mathbf{B}} = \frac{1}{\sqrt{30}} \begin{pmatrix} \sqrt{5} & \sqrt{5} & 2\sqrt{5} \\ 2\sqrt{6} & 0 & -\sqrt{6} \\ -1 & 5 & -2 \end{pmatrix}, \quad \hat{\mathbf{T}} = \begin{pmatrix} 2\sqrt{5} & 7\sqrt{5} & 10\sqrt{5} \\ 12\sqrt{6} & 0 & -9\sqrt{6} \\ 4 & 5 & 2 \end{pmatrix}.$$

Calculate the matrix $\hat{\mathbf{U}}$ of the operator $\mathbf{U}$, and show that the operator $\mathbf{U}$ is positive-definite. ***(13 marks)***

2 The motion of a continuum is given by

$$x_1 = \xi_1 + \sqrt{3}\omega^2 t^2 \xi_3, \quad x_2 = \xi_2 \quad x_3 = (2 + \omega^2 t^2)\xi_3,$$

where ω is a constant, and the vectors $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\boldsymbol{x} = (x_1, x_2, x_2)$ determine the initial and current position of a point, respectively.

(i) Calculate the velocity and acceleration. ***(4 marks)***

(ii) Initially the continuum density is ρ_0. Calculate the continuum density ρ at current time, t. ***(6 marks)***

(iii) The plane Π_0 is initially given by the equation $\xi_1 = a$, where a is a constant.

(a) Show that at time t this plane deforms onto another plane Π_t given by

$$x_1 = a + \frac{\sqrt{3}\,\omega^2 t^2 x_3}{2 + \omega^2 t^2},$$

and that for any point $B \in \Pi_t$ there is a unique point $A \in \Pi_0$ such that A is mapped to B. ***(9 marks)***

(b) Find the angle between Π_0 and Π_t and determine the limiting value of this angle as $t \to \infty$. ***(6 marks)***

3 (i) Write down the expression for the surface traction, $\boldsymbol{t}$, in terms of the stress tensor, $\mathbf{T}$, and the unit normal to the surface, $\boldsymbol{n}$. Express it both in vector and coordinate form. ***(3 marks)***

(ii) You are given that, in the absence of body forces, the equilibrium equation in the integral form is given by

$$\int_S \boldsymbol{t}\, dS = 0,$$

where S is an arbitrary closed surface. Assuming that the components of the stress tensor have continuous derivatives, derive from this equation the equilibrium equation in the differential form in Cartesian coordinates,

$$\frac{\partial T_{ij}}{\partial x_j} = 0, \quad i = 1, 2, 3,$$

where T_{ij} are the components of the stress tensor.

[You can use without proof that, if the function $f(\boldsymbol{x})$ is continuous and $\int_V f(\boldsymbol{x})\, dV = 0$ for any volume V, then $f(\boldsymbol{x}) \equiv 0$.] ***(9 marks)***

(iii) The stress components in a plate bounded by $x_1 = \pm l$ and $x_2 = \pm h$ are given by

$$T_{11} = wn^2 \sin(\pi x_1/2l) \cosh(nx_2),$$
$$T_{22} = -A \sin(\pi x_1/2l) \cosh(nx_2),$$
$$T_{12} = -B \cos(\pi x_1/2l) \sinh(nx_2),$$
$$T_{13} = T_{23} = T_{33} = 0$$

where w, n, A, and B are constants.

(a) You are given that the plate is in equilibrium. Express the constants A and B in terms of w and n. ***(6 marks)***

(b) Find the stress vectors on the edges $x_1 = l$ and $x_2 = -h$. ***(7 marks)***

4 Fluid motion is called potential when $\boldsymbol{v} = \nabla\Phi$, where Φ is the velocity potential. In the case of an ideal incompressible fluid, the velocity potential satisfies the Laplace equation $\nabla^2\Phi = 0$.

Consider a stationary potential flow of an incompressible fluid near a rigid sphere. The sphere surface is defined by the equation $x^2 + y^2 + z^2 = a^2$ in Cartesian coordinates x, y, z. You are given that the flow is axisymmetric, i.e. it is independent of φ in spherical coordinates r, θ, φ with the origin at the centre of the sphere, and the angle θ measured from the positive x-axis. You are also given that the flow velocity has the magnitude V and it is in the negative x-direction far from the sphere.

(i) Write down the boundary condition for the velocity at the sphere surface in terms of Φ. ***(3 marks)***

(ii) Find the velocity potential Φ in spherical coordinates.

[You can use the expression for the Laplace operator in spherical coordinates, $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$.]

Hint: Look for the solution of the Laplace equation in the form $\Phi = R(r)\cos\theta$, where $R(r)$ is the function to be determined. Also look for solutions to the equation for $R(r)$ in the form $R(r) = r^\alpha$, where α is to be determined. ***(12 marks)***

(iii) Use Bernoulli's integral for the fluid stationary motion,

$$p + \frac{\rho}{2}\|\mathbf{v}\|^2 = p_0,$$

where p and ρ are the fluid pressure and density, and p_0 is a constant, to calculate the pressure at the surface of the sphere. ***(4 marks)***

(iv) Calculate the total pressure force acting on the sphere, and thus prove the d'Alembert paradox: The total force acting on the sphere is zero.

[You can use the expression for the integral over the surface of the sphere: $\int_S f\,dS = a^2\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} f(\theta,\varphi)\,d\varphi$.] ***(6 marks)***

5 (i) You are given that, for linear elasticity, the equation of motion is given by

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \nabla \cdot \mathbf{T} + \rho_0 \boldsymbol{b}, \qquad (*)$$

where $\boldsymbol{u}$ is the displacement, $\boldsymbol{b}$ the body force, and ρ_0 the density. You are also given that, in an isotropic material, the Cartesian components of the stress tensor are given by

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where u_i are the Cartesian components of the displacement $\boldsymbol{u}$, and λ and μ are the Lamé constants. Show that, when $\boldsymbol{b} = 0$, equation $(*)$ reduces to

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u}. \qquad (\dagger)$$

(6 marks)

(ii) There is an elastic spherical shell of internal radius R and external radius $R + l$. You are given that the displacement in the shell is in the radial direction and only depends on the distance from the sphere centre. Hence, in the spherical coordinates, $\boldsymbol{u} = u(t, r)\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit vector in the radial direction. Show that, in this case, the equation of motion reduces to

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2u}{r^2} \right). \qquad (\ddagger)$$

$\Bigg[$You can use without proof that $\nabla^2 (u(t, r)\boldsymbol{e}_r) = \nabla[\nabla \cdot (u(t, r)\boldsymbol{e}_r)]$ and $\nabla \cdot (u(t, r)\boldsymbol{e}_r) = \dfrac{1}{r^2} \dfrac{d(r^2 u)}{dr}\Bigg]$.

(6 marks)

(iii) You are now given that $l \ll R$, so the terms proportional to u and its first derivative with respect to r can be neglected relative to the term proportional to the second derivative. As a result, equation $(\ddagger)$ reduces to the wave equation

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial r^2}.$$

Search for solutions to this equation in the form $u = \sin(\omega t) U(r)$, and find the general expression for $U(r)$. ***(4 marks)***

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5 (continued)

(iv) You are given that the surface tractions at the shell boundaries are given by

$$\boldsymbol{t}(a) = \left(\frac{\lambda}{r^2} \frac{d(r^2 u)}{dr} \bigg|_{r=a} + 2\mu \frac{du}{dr} \bigg|_{r=a} \right) \boldsymbol{n} \approx (\lambda + 2\mu) \frac{du}{dr} \bigg|_{r=a} \boldsymbol{n},$$

where $a = R$ and $\boldsymbol{n} = -\boldsymbol{e}_r$ at the internal boundary, and $a = R + l$ and $\boldsymbol{n} = \boldsymbol{e}_r$ at the external boundary. Use the condition of continuity of the surface traction at the shell boundaries to calculate all possible positive values of the oscillation frequency ω for the solution of part (iii).

[You can neglect the air pressure at take $\boldsymbol{t}(a) = 0$.]

Hint: Recall the condition when a system of two linear homogeneous algebraic equations has non-trivial solutions. ***(9 marks)***

End of Question Paper