The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015-16**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$ (where $i^2 = -1$):

(a) $J_1 = \{a + bi\sqrt{11} : \ a, b \in \mathbb{Q}\}$, *(4 marks)*

(b) $J_2 = \{a + bi + ci\sqrt{-4} : \ a, b, c \in \mathbb{Q}\}$. *(3 marks)*

(ii) Let $L = \mathbb{Q}(\sqrt{5}, i)$ and $a = \sqrt{5} + i$.

(a) Find $[L : \mathbb{Q}]$. Justify your response. *(5 marks)*

(b) Show that $L = \mathbb{Q}(a)$. *(5 marks)*

(c) Express the element $\dfrac{1}{1 + \sqrt{5} + i}$ in the form

$$\alpha + \beta\sqrt{5} + \gamma i + \delta i\sqrt{5}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$. *(5 marks)*

(iii) State Eisenstein's Irreducibility Criterion. *(3 marks)*

2 (i) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. *(2 marks)*

(ii) Give a proof of the degrees formula. *(9 marks)*

(iii) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{2}i)$ and $b = \sqrt{2}i$.

(a) Find a $\mathbb{Q}$-basis of the field L. *(5 marks)*

(b) Find the minimal polynomial of the element b over the field $\mathbb{Q}$. *(5 marks)*

(c) Is $L = \mathbb{Q}(b)$? Justify your response. *(4 marks)*

3 (i) Let L and M be subfields of the fields of complex numbers $\mathbb{C}$ such that $n = [L : \mathbb{Q}]$ and $m = [M : \mathbb{Q}]$ are co-prime natural numbers (i.e. the greatest common divisor of n and m is 1). Let LM be the subfield of $\mathbb{C}$ generated by the fields L and M. What is $[LM : \mathbb{Q}]$? Justify your response. ***(7 marks)***

(ii) Define the content of a polynomial and find the content of the product of polynomials $p = p_1 p_2 \cdots p_{10}$ where $p_i = \sum_{j=i}^{10} 2^j x^n$. ***(5 marks)***

(iii) Let $f, g \in \mathbb{Z}[x]$ be nonzero polynomials with integer coefficients. What is the content $c(fg)$ of their product? Justify your response (there is no need to prove Gauss's Lemma). ***(6 marks)***

(iv) Let $f \in \mathbb{Z}[x]$ be a primitive polynomial. Prove that if the polynomial f is reducible in $\mathbb{Q}[x]$ then $f = pq$ for some non-constant polynomials $p, q \in \mathbb{Z}[x]$. ***(7 marks)***

4 (i) Define the n'th cyclotomic polynomial ϕ_n. Give an explicit expression for the cyclotomic polynomial where $n = p$ is a prime number. ***(5 marks)***

(ii) Show that $x^n - 1 = \prod_{l|n} \phi_l$. ***(5 marks)***

(iii) Define a quadratic field extension L/K. Which of the field extensions L_1 and L_2 of the field of rational numbers $\mathbb{Q}$ are quadratic, where $L_1 = \mathbb{Q}(1+2\sqrt{-2})$ and $L_2 = \mathbb{Q}(\sqrt[3]{7})$? Justify your response. ***(9 marks)***

(iv) State a criterion for a point $P = (a, b) \in \mathbb{R}^2$ to be a constructible point. Using the criterion (or otherwise) decide whether the point $P = (1+\sqrt[4]{3}, 2+3\sqrt{41})$ is constructible or not. ***(6 marks)***

End of Question Paper