Data provided: list of formulae

MAS336

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

Differential Geometry **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

A list of formulae is provided on the last three pages.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Below is a list of *true/false* and *multiple answers* questions. Each correct answer is worth 2 marks.

(1) How many reparametrisations of $\gamma\colon \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$ exist?

(a) None at all.

(b) 56.

(c) Uncountably many.

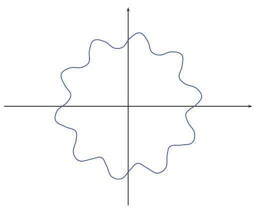
(d) Exactly one.

(4 marks)

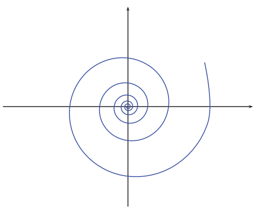
(2) *The curve* $]0, +\infty[\to \mathbb{R}$, $t \mapsto (t^2, t^3)$ *is regular.* Is this true? false? ***(4 marks)***

(3) *A parametrised surface can be smooth and not regular.* Is this true? false? ***(4 marks)***

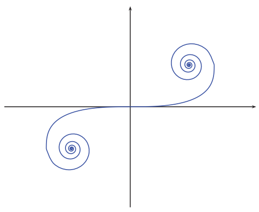
(4) Which of the following pictures could illustrate a unit-speed parametrised curve whose curvature is $\kappa\colon \mathbb{R} \to \mathbb{R}$, $\kappa(t) = t$? ***(5 marks)***



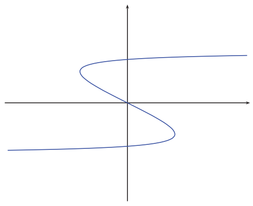
Curve (a)



Curve (b)



Curve (c)



Curve (d)

(5) Consider a point $P = \sigma(u, v)$ on a parametrised surface S. Assume that the Weingarten matrix $W_{(u,v)}$ has negative (nonzero) determinant. Then, locally around P, the surface looks like

(a) a saddle;

(b) a bowl;

(c) it is not possible to predict that without knowing the eigenvalues of $W_{(u,v)}$.

(4 marks)

(6) *The torus can be triangulated with* 4 *edges,* 5 *faces and* 2 *vertices.* Is this true? false? ***(4 marks)***

2 (i) Consider the two curves $\gamma_1\colon\ ]1,+\infty[\ \to \mathbb{R}^2$, $\gamma_2\colon\ ]1,+\infty[\ \to \mathbb{R}^2$,

$$\gamma_1(t) = \big(\cos t + t\sin t, \sin t - t\cos t\big), \qquad \gamma_2(t) = \big(\ln(\sqrt{t} + \sqrt{t-1}), \sqrt{t}\big).$$

(a) What does the classification of plane curves say? ***(4 marks)***

(b) Compute the first and second derivatives of γ_1. ***(2 marks)***

(c) Compute the first derivative of γ_2 and show that its second derivative is given by

$$\ddot{\gamma}_2(t) = -\frac{1}{4}\left(\frac{2t-1}{(t(t-1))^{3/2}}, \frac{1}{t^{3/2}}\right).$$

(3 marks)

(d) Show that the curvature functions of γ_1 and γ_2 are equal by directly computing each. ***(3 marks)***

(e) Does it follow that the curves are congruent? Justify your answer. ***(3 marks)***

(ii) (a) Assume that we know the curvature function $\kappa\colon \mathbb{R} \to \mathbb{R}$ of a unit-speed plane curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ and the values $\gamma(0) = \mathbf{0}$ and $\dot{\gamma}(0) = (1,0)$. Explain why $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ is then given by

$$\gamma(t) = \int_0^t (\cos\theta(u), \sin\theta(u))du$$

with $\theta(t) = \displaystyle\int_0^t \kappa(u)du$. ***(4 marks)***

(b) Give an equation of the unit-speed parametrised curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ with curvature function $\kappa\colon \mathbb{R} \to \mathbb{R}$, $\kappa(t) = 5$ and with $\gamma(0) = \mathbf{0}$ and $\dot{\gamma}(0) = (1,0)$. Describe its image. ***(6 marks)***

3 The hyperboloid $x^2 + y^2 - z^2 = 25$ can be parametrised by

$$\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad (z,\theta) \mapsto (\sqrt{z^2+25}\cos\theta, \sqrt{z^2+25}\sin\theta, z).$$

(i) In a few sentences or a sketch, explain geometrically the parametrisation, and give the name of this general type of parametrised surfaces. ***(2 marks)***

(ii) Explain what it means for a parametrised surface to be smooth and regular and check that σ is smooth and regular. ***(6 marks)***

(iii) Find an expression for a unit normal to the surface. ***(2 marks)***

(iv) Find an equation for the tangent plane to the surface at $(x_0, y_0, 0) = \sigma(\theta_0, 0)$. ***(4 marks)***

(v) The two lines $(x_0, y_0, 0) + t(-y_0, x_0, 5)$ and $(x_0, y_0, 0) + t(y_0, -x_0, 5)$ lie in the surface and in the tangent plane found in (iv). Prove this for the first line, $(x_0, y_0, 0) + t(-y_0, x_0, 5)$. ***(4 marks)***

(vi) The second derivatives of σ are given by

$$\sigma_{zz}(z,\theta) = \frac{25}{(z^2+25)^{\frac{3}{2}}}(\cos\theta, \sin\theta, 0),$$

$$\sigma_{\theta z}(z,\theta) = \frac{z}{\sqrt{z^2+25}}(-\sin\theta, \cos\theta, 0)$$

and

$$\sigma_{\theta\theta}(z,\theta) = -\sqrt{z^2+25}\,(\cos\theta, -\sin\theta, 0).$$

Compute the principal curvatures and directions and the Gaussian curvature at $(x_0, y_0, 0)$. What is the geometric meaning of the two lines in (v)? ***(7 marks)***

4 (i) The torus of revolution is the surface parametrised by $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3$,

$$\sigma(t,\theta) = ((R + r\cos t)\cos\theta, (R + r\cos t)\sin\theta, r\sin t)$$

with $0 < r < R$. Compute the area of the image of this parametrisation, the torus. ***(8 marks)***

(ii) Find a point on the torus with negative Gaussian curvature, and one point with positive Gaussian curvature. State the principal directions and curvatures at these points. Your answers do not need to be computational and can be explained with figures. ***(8 marks)***

(iii) Consider a surface that is isometrically parametrised.

(a) Using the Gauss equations in the list of formulae, prove that the Gaussian curvature is 0 everywhere. ***(5 marks)***

(b) What can you deduce about the principal curvatures? ***(2 marks)***

(c) Deduce from this that the torus cannot be parametrised isometrically. ***(2 marks)***

5 (i) Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^3$ and $P \in \mathbb{R}^3$ a point. Parametrise the plane in $\mathbb{R}^3$ through the point P and with directions $\mathbf{u}$ and $\mathbf{v}$. ***(2 marks)***

(ii) Compute the first fundamental form of this parametrised plane. When does the parametrisation preserve arc-lengths? angles? areas? Describe the geometric properties that the pair of vectors $\mathbf{u}$ and $\mathbf{v}$ must have in each of the cases. ***(8 marks)***

(iii) Using (ii), find an example of a parametrisation of a surface that preserves angles, but does not preserve distances and does not preserve surface areas. Find an example of a parametrisation of a surface that preserves surface areas, but does not preserve angles and does not preserve distances. ***(4 marks)***

(iv) Find an example of a parametrisation of a plane that preserves angles and stretches arc-lengths by a factor 2. Justify your answer. ***(6 marks)***

(v) Give an example of a parametrised surface (not a plane) that is locally isometric to the Euclidean plane. Justify your answer. ***(5 marks)***

End of Question Paper

List of Formulae

- The inverse of a 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{R}$ and $ad - bc \neq 0$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- The cross-product of 2 vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is

$$v_1 \times v_2 = (y_1 z_2 - z_1 y_2, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1) \in \mathbb{R}^3.$$

- The angle θ between two vectors v_1 and $v_2 \in \mathbb{R}^3$ is given by

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}.$$

For a curve on $\mathbb{R}^2$ parametrised by $\gamma\colon (\alpha, \beta) \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$:

- The arc length from $\gamma(a)$ to $\gamma(b)$, $\alpha < a \leq b < \beta$ is:

$$\int_a^b ||\dot\gamma(t)|| \, dt$$

- The curvature of γ at $\gamma(t)$ is

$$\kappa(t) = \frac{\ddot\gamma(t) \cdot J(\dot\gamma(t))}{\|\dot\gamma(t)\|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}},$$

  where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the counterclockwise rotation of angle $\pi/2$.

For a parametrised surface $\sigma\colon U \to \mathbb{R}^3$, with U an open set in $\mathbb{R}^2$:

- The first fundamental form is given by

$$\mathrm{I}_{(u,v)} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$$

  for all $(u, v) \in \mathbb{R}^2$, with $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$ and $G = \sigma_v \cdot \sigma_v$.

- Area of the domain $\sigma([\alpha_1, \beta_1] \times [\alpha_2, \beta_2])$, for $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subseteq U$:

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \sqrt{EG - F^2}\, dv\, du$$

- The preferred unit normal vector along σ is given by $\mathbf{n} \colon U \to \mathbb{R}^3$,

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}.$$

- The second fundamental form of σ at $(u, v) \in U$ is

$$\mathrm{II}_{(u,v)} = \begin{pmatrix} L(u,v) & M(u,v) \\ M(u,v) & N(u,v) \end{pmatrix}$$

where $L = \sigma_{uu} \cdot \mathbf{n}$, $M = \sigma_{uv} \cdot \mathbf{n}$ and $N = \sigma_{vv} \cdot \mathbf{n}$.

- The Weingarten matrix of σ is

$$W = \mathrm{I}^{-1}\, \mathrm{II}\,.$$

- The Gaussian curvature is

$$K = \det W.$$

The Gauss equations:

$$\begin{aligned}
EK &= (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2 \\
FK &= (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} \\
FK &= (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11} \\
GK &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22}.
\end{aligned}$$

where

$$\begin{aligned}
&\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \quad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\
&\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}, \quad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)}, \\
&\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.
\end{aligned}$$

Gauss-Bonnet formula for a geodesic triangle Δ on a surface:

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \iint_\Delta K dA$$

where $\alpha_1, \alpha_2, \alpha_3$ are the interior angles of Δ.

Gauss-Bonnet formula for a compact surface S:

$$\iint_S K dA = 2\pi\chi(S).$$