Data provided: Formula sheet

The University Of Sheffield.

MAS380

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2015-2016

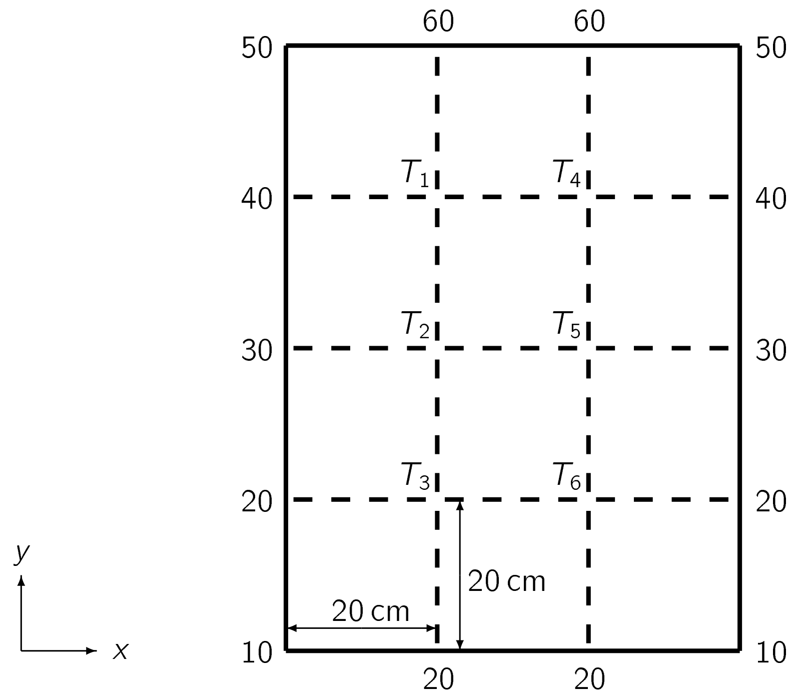
Computational Engineering Mathematics

Three hours

Marks will be awarded for your best FOUR answers

1 The figure shows a rectangular plate made of a homogeneous isotropic material. The plate is divided into intervals of equal length 20 cm in the x and y directions. The temperature $T(x, y)$ in this plate satisfies the indicated boundary conditions (given in °C) and has reached a steady-state condition so that it is described by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$



(a) Draw a sketch of the solution domain, showing clearly the line of symmetry for $T(x, y)$, and indicating which of the unknown temperatures $T_1, \ldots, T_6$ are equal to each other. ***(5 marks)***

(b) Use the finite difference formulae on the formula sheet to find the finite difference equations required to find estimates of the nodal temperatures T_1, T_2 and T_3. ***(10 marks)***

(c) Express the finite difference equations in part (b) in the form $A\mathbf{T} = \mathbf{b}$, where A is a 3×3 matrix, $\mathbf{T} = (T_1, T_2, T_3)^T$ and $\mathbf{b} = (100, 30, 40)^T$, with units in °C, where you should give the matrix A.

Hence, using Gaussian elimination or otherwise, show that

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 310 \\ 230 \\ 170 \end{bmatrix} \text{°C}.$$ ***(10 marks)***

2 The temperature $T(x, t)$ satisfies the convection-diffusion equation

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} \qquad (0 \leqslant x \leqslant 1). \tag{1}$$

(a) If $T_{i,j} = T(x_i, t_j)$, with $i = 0$ and $i = N$ corresponding to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponding to $t = 0$, use backward differences for time derivatives and central differences for space derivatives to derive the implicit scheme

$$-(k-\beta)T_{i+1,j} + (1+2k)T_{i,j} - (k+\beta)T_{i-1,j} = T_{i,j-1}$$

for $i = 1, \ldots, N-1$ and $j = 1, 2, \ldots,$ where

$$k = \frac{\Delta t}{(\Delta x)^2} \quad \text{and} \quad \beta = \frac{\Delta t}{2\Delta x}.$$ ***(5 marks)***

(b) Equation (1) is to be solved (approximately) over $0 \leqslant x \leqslant 1$, with boundary conditions $T(0, t) = 20$ and $T(1, t) = 30$, and initial temperature distribution $T(x, 0) = 20 + 10x$.

Taking $\Delta x = 0.25$ and $\Delta t = 0.1$, use the implicit scheme in part (a) to write down the system of equations for the temperature at $x = 0.25, 0.5, 0.75$ and time $t = 0.1$. (Note that you **do not** need to solve the equations.) ***(12 marks)***

(c) Show from your answer to part (b) that the Jacobi iteration equations to find the $(k+1)$th iteration from the kth iteration are

$$\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \end{bmatrix}^{(k+1)} = \frac{1}{4.2}\begin{bmatrix} 58.5 \\ 25 \\ 69.5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{3}{7} & 0 & \frac{1}{3} \\ 0 & \frac{3}{7} & 0 \end{bmatrix} \begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \end{bmatrix}^{(k)}.$$ ***(8 marks)***

3 You are given that

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl},$$

where σ_{ij} is the stress tensor, ε_{ij} is the (symmetric) strain tensor, and C_{ijkl} is a fourth order tensor of constant coefficients. In the following, δ_{ij} is the Kronecker delta tensor.

(a) Given that

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})$$

for constants λ and μ, show that

$$\sigma_{ij} = \lambda\,\delta_{ij}\,\varepsilon_{mm} + 2\mu\,\varepsilon_{ij}.$$ ***(5 marks)***

Show that the mean normal stress $\frac{1}{3}\sigma_{kk}$ is

$$\frac{1}{3}\sigma_{kk} = \left(\lambda + \frac{2}{3}\mu\right)\varepsilon_{mm}.$$ ***(3 marks)***

(b) Given that $\lambda = 0.6391\,\mathrm{Pa}$ and $\mu = 0.9068\,\mathrm{Pa}$, and that the strain tensor satisfies $\varepsilon_{11} = -0.3014\times10^{-2}$, $\varepsilon_{22} = 0.5315\times10^{-2}$, $\varepsilon_{33} = 0.4353\times10^{-2}$, $\varepsilon_{12} = 0.3124\times10^{-2}$, $\varepsilon_{23} = 0.7518\times10^{-2}$, and $\varepsilon_{31} = -0.5416\times10^{-2}$, find the stress tensor. ***(11 marks)***

Hence find the stress force on a unit area element in the direction $\mathbf{n} = (0.5774, -0.5774, 0.5774)^T$. ***(6 marks)***

4 The velocity field in a fluid is given by $\mathbf{v} = u\,\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, and the density by ρ, where u, v, w and ρ are functions of x, y, z and t.

(a) The divergence of $\mathbf{v}$ is defined by

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1}{\Delta\mathcal{V}}\frac{D}{Dt}(\Delta\mathcal{V}),$$

where $\Delta\mathcal{V}$ is an infinitesimal control volume and $\dfrac{D}{Dt}$ is the substantial derivative. By considering the mass of the moving control volume, $\Delta m = \rho\Delta\mathcal{V}$, derive the equation of continuity (i.e. of mass conservation), and show that it can be written as

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0.$$

(8 marks)

(b) Let $x = x_1$, $y = x_2$ and $z = x_3$.

(i) The i component of the curl of a vector $\mathbf{u}$, $\nabla \times \mathbf{u}$, can be written using index notation as

$$(\nabla \times \mathbf{u})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}u_k,$$

where ε_{ijk} is the Levi-Civita tensor.

Show that

$$\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_k}\right) = 0,$$

i.e.

$$\nabla \times \nabla\phi = \mathbf{0},$$

for any ϕ. ***(4 marks)***

(ii) The vorticity $\boldsymbol{\omega}$ is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}, \qquad \text{i.e. } \omega_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}v_k.$$

Show that

$$(\mathbf{v}\times\boldsymbol{\omega})_i = \frac{\partial}{\partial x_i}\left(\frac{1}{2}v_j v_j\right) - v_j\frac{\partial}{\partial x_j}v_i.$$

(4 marks)

[You may assume that $\varepsilon_{kij}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$.]

(iii) The momentum equation may be written as

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial\sigma_{ji}}{\partial x_j} + \frac{1}{\rho}F_i,$$

where p is the pressure, σ_{ij} is the stress tensor, and $\mathbf{F}$ is the body force.

 Question 4 continued on next page

4 (continued)

Show that the momentum equation can be rewritten as

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}v_j v_j\right) - (\mathbf{v}\times\boldsymbol{\omega}\ )_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ji}}{\partial x_j} + \frac{1}{\rho}F_i, \qquad (2)$$

and show that

$$[\nabla\times(\mathbf{v}\times\boldsymbol{\omega}\ )]_i = v_i\frac{\partial \omega_j}{\partial x_j} + \omega_j\frac{\partial v_i}{\partial x_j} - v_j\frac{\partial \omega_i}{\partial x_j} - \omega_i\frac{\partial v_j}{\partial x_j}.$$ ***(5 marks)***

[Again you may assume that $\varepsilon_{kij}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$.]

Hence, by taking the curl of (2), show that

$$\frac{\partial \omega_i}{\partial t} + v_j\frac{\partial \omega_i}{\partial x_j} = \omega_j\frac{\partial v_i}{\partial x_j}$$

if the density ρ is constant, the body force is conservative (i.e. $\nabla\times\mathbf{F} = \mathbf{0}$) and the flow is incompressible (i.e. $\nabla\cdot\mathbf{v} = 0$).

(4 marks)

[You may assume that $\nabla\cdot(\nabla\times\mathbf{u}) = 0$ for any $\mathbf{u}$.]

5 A one-dimensional bar has variable cross-sectional area $A(x)$, is fixed at one end ($x = 0$) and is loaded axially by a known force F_ℓ at its free end ($x = \ell$). The total force acting across the cross-sectional area located at x is denoted by $F(x)$. The body force per unit length is $f(x)$.

(a) The strong form of the one-dimensional equation of force balance is

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + f = 0,$$

where $u(x)$ is the displacement in the x direction (so the strain $\varepsilon_{xx} = du/dx$) and the stress $\sigma_{xx} = E\varepsilon_{xx}$, where E is a constant. The boundary conditions are $u(0) = 0$ and $F(\ell) = F_\ell$.

By multiplying this by an arbitrary weighting function $w(x) > 0$, derive the weak form of the equation:

$$\int_0^\ell \frac{dw}{dx} A\, E \frac{du}{dx} dx = w(\ell)F(\ell) - w(0)F(0) + \int_0^\ell w\, f\, dx\,.$$ ***(7 marks)***

(b) Consider a solution domain $0 \leqslant x \leqslant \ell$, with two nodes (at $x_1 = 0$ and $x_2 = \ell$). The trial solution is $u(x) \approx U_1 N_1^1(x) + U_2 N_2^1(x) = \mathbf{N}^T\mathbf{U}$ where $\mathbf{N} = (N_1^1, N_2^1)^T$, $\mathbf{U} = (U_1, U_2)^T$, and U_1 and U_2 are constants. The weight function is $w(x) = c_1 N_1^1(x) + c_2 N_2^1(x) = \mathbf{c}^T\mathbf{N}$, where $\mathbf{c} = (c_1, c_2)^T$ and c_1 and c_2 are arbitrary constants. N_1^1 and N_2^1 are defined by

$$N_1^1(x) = \frac{x_2 - x}{x_2 - x_1}, \qquad N_2^1(x) = \frac{x - x_1}{x_2 - x_1}.$$

(i) Show that

$$E\left[\int_0^\ell A \frac{d\mathbf{N}}{dx}\frac{d\mathbf{N}}{dx}^T dx\right]\mathbf{U} = \mathbf{N}(\ell)F(\ell) - \mathbf{N}(0)F(0) + \int_0^\ell \mathbf{N}\, f\, dx.$$

(4 marks)

(ii) Given that A and f are constant, obtain the pair of simultaneous finite element equations

$$U_1 - U_2 = \frac{\ell}{AE}\left(\frac{\ell f}{2} - F(0)\right), \quad U_2 - U_1 = \frac{\ell}{AE}\left(\frac{\ell f}{2} + F(\ell)\right).$$

(9 marks)

Given that $F(0) = \ell f + F(\ell)$, deduce that

$$u(x) \approx \frac{x}{AE}\left(\frac{\ell f}{2} + F(\ell)\right).$$ ***(5 marks)***

End of Question Paper

Formula Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j+1} - U_{i,j}}{\Delta t}$$

Forward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i,j}}{\Delta x}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta t}$$

Backward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2}(x_i,\ t_j) \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2}$$

Relation between different parameters:

A number of relationships between E, ν, K, λ and μ hold and are summarized in Table 1. μ ($\equiv G$) is the elastic shear modulus, K the elastic bulk modulus, E the elastic stiffness (or Young's Modulus) and ν Poisson's ratio.

	E	ν	K	λ	$\mu \equiv G$
E, ν	-	-	$\frac{E}{3(1-2\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$
E, K	-	$\frac{3K-E}{6K}$	-	$\frac{K(9K-3E)}{9K-E}$	$\frac{3KE}{9K-E}$
K, μ	$\frac{9\mu K}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	-	$K-\frac{2\mu}{3}$	-

Table 1: The relations between the properties of elastic bodies.