Data provided: Formula sheet

MAS381

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

Mathematics III (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let us write a complex function as $z = x + jy$, where x and y are real. Sketch the four regions in the z-plane corresponding to $x \geq 1$, $y \geq x + 4$, $|z| \leq 3$ and $|z + 2| \leq 1$. ***(8 marks)***

(ii) For the mapping $w = u + jv = (4 + 3j)z + j$ (here u, v are real functions),

(a) find $u(x, y)$ and $v(x, y)$;

(b) find the image in the w-plane of the region $|z| \leq 1$ in the z-plane;

(c) sketch both the region $|z| \leq 1$ and the image of this region in the w-plane. ***(13 marks)***

(iii) Find out if the image of the circle $|z| = \sqrt{2}$ under the bilinear mapping $w = \dfrac{4z + 1 - 3j}{z - 1 - j}$ is a circle or a straight line. ***(4 marks)***

2 (i) Find the first three non-zero terms of the power series expansion of the function $f(z) = \dfrac{z}{(z + j)(z - 2j)}$ about the point $z = 0$. Show the region of convergence of the power series and all poles and zeros of $f(z)$ on the Argand diagram. ***(14 marks)***

(ii) Expand the function $f(z)$ defined above into Laurent series in the annulus $1 < |z| < 2$. ***(11 marks)***

3 (i) Find all the poles and zeros of $f(z) = \dfrac{z}{(9-z^2)(z+j)}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)dz$, writing your solutions in the form $a + jb$, where a and b are real, and

(a) C is the circle $|z+3| = 5$.

(b) C is the circle $|z| = \dfrac{1}{2}$

(16 marks)

(ii) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2 - 4x + 5} dx.$$

(9 marks)

4 (i) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (2x + z^2)\mathbf{i} + xz\mathbf{j} + 3xy\mathbf{k}$, and C is the curve from $(0,0,0)$ to $(1,1,2)$ given in parametric form by $\mathbf{r} = t^2\mathbf{i} + t^3\mathbf{j} + 2t\mathbf{k}$, where the parameter t has the range $0 \leq t \leq 1$.

(8 marks)

(ii) Define Stoke's theorem and verify it for the vector function

$$\mathbf{F} = y^2\mathbf{i} - (x+z)\mathbf{j} + yz\mathbf{k}$$

and the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$, for $z = 0$. ***(17 marks)***

End of Question Paper

Formula sheet

- The general formula for the residue at a pole z_0, of order m is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

- Useful identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}; \quad \sin m\theta\cos n\theta = \frac{1}{2}[\sin(m+n)\theta + \sin(m-n)\theta]$$