The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS   Autumn Semester 2015–2016

Topics in Advanced Fluid Mechanics   **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Let $\boldsymbol{v}$ be the velocity, p the pressure, ρ the (constant) density, and $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ the vorticity. Use the Euler equation for an ideal incompressible fluid written in the Gromeka-Lamb form,

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{v}|^2\right),$$

to derive the equation

$$\frac{\partial |\boldsymbol{v}|^2}{\partial t} = -2\nabla\cdot\left[\boldsymbol{v}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{v}|^2\right)\right].$$

Assuming that $\boldsymbol{v}$ tends to zero faster than r^{-2} as $r\to\infty$, where r is the distance from the coordinate origin, prove the helicity conservation law,

$$\int_{R^3} |\boldsymbol{v}|^2\, dV = \text{const.}$$

Hint: $\nabla\cdot(f\boldsymbol{a}) = \boldsymbol{a}\cdot\nabla f + f\nabla\cdot\boldsymbol{a}$. ***(6 marks)***

(ii) Using the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v}$$

show that the helicity $\boldsymbol{v}\cdot\boldsymbol{\omega}$ satisfies the equation

$$\frac{\partial(\boldsymbol{v}\cdot\boldsymbol{\omega})}{\partial t} = \nabla\cdot\left[\boldsymbol{\omega}\left(\frac{1}{2}|\boldsymbol{v}|^2 - \frac{p}{\rho}\right) - \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\omega})\right]. \qquad (*)$$

Hint: $\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot\nabla\times\boldsymbol{a} - \boldsymbol{a}\cdot\nabla\times\boldsymbol{b}$. ***(7 marks)***

Question 1 continued on next page

1 (continued)

(iii) The impulse density is defined by $\boldsymbol{\gamma} = \boldsymbol{v} + \nabla\varphi$. You are given that $\boldsymbol{\gamma}$ and φ satisfy the system of equations

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla\boldsymbol{v})^T \cdot \boldsymbol{\gamma}, \quad \frac{D\varphi}{Dt} = \frac{p}{\rho} - \frac{1}{2}|\boldsymbol{v}|^2,$$

where the components of tensor $\nabla\boldsymbol{v}$ are $(\nabla\boldsymbol{v})_{ij} = \partial v_i/\partial x_j$ and 'T' indicates the transposed tensor.

(a) Calculating the i-component of the left- and right-hand sides prove the identity

$$\frac{D\nabla\varphi}{Dt} = \nabla\frac{D\varphi}{Dt} - (\nabla\boldsymbol{v})^T \cdot \nabla\varphi.$$

(3 marks)

(b) Use this identity and equation $(*)$ to show that

$$\frac{D(\boldsymbol{\gamma}\cdot\boldsymbol{\omega})}{Dt} = 0.$$

(9 marks)

2 Burgers' equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

(i) Introduce the velocity potential $u = \dfrac{\partial \phi}{\partial x}$ to reduce Burgers' equation to

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \nu\frac{\partial^2 \phi}{\partial x^2}.$$

Hint: If ϕ is a velocity potential, then $\tilde{\phi} = \phi + f(t)$, where $f(t)$ is an arbitrary function, is also a velocity potential. ***(5 marks)***

(ii) Put $\phi = -2\nu \log \psi$. Show that ψ satisfies the heat conduction equation

$$\frac{\partial \psi}{\partial t} = \nu\frac{\partial^2 \psi}{\partial x^2}. \qquad (*)$$

(5 marks)

(iii) (a) You are given that, at the initial time ($t = 0$), $\psi = \psi_0(x) \equiv e^{|x|/2\nu}$. Calculate the expression for u at the initial time. ***(3 marks)***

(b) Calculate the limiting expression for ψ as $t \to \infty$. Then find the limiting expression for u.

Hint: The solution to the initial value problem for the heat conduction equation $(*)$ is given by

$$\psi(t,x) = \frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{\infty} \psi_0(y)\exp\left(-\frac{(x-y)^2}{4\nu t}\right) dy.$$

You also can use the formula $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \pi.$ ***(12 marks)***

3 Consider the equation

$$\frac{\partial \omega}{\partial t} = e^{-t}\omega\mathcal{H}[\omega] - \gamma\omega,$$

where γ is a constant, $\gamma > -1$, and

$$\mathcal{H}[\omega] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\omega(y)}{x-y}dy$$

is the Hilbert transform and $\mathcal{P}$ indicates the principal Cauchy part of integral.

(i) Show that the variable substitution

$$\tau = \frac{1 - e^{-(\gamma+1)t}}{\gamma + 1}, \quad \Omega = e^{\gamma t}\omega$$

reduces this equation to

$$\frac{\partial \Omega}{\partial \tau} = \Omega\mathcal{H}[\Omega]. \qquad (*)$$

(4 marks)

(ii) Derive the equation

$$\frac{\partial}{\partial \tau}\mathcal{H}[\Omega] = \frac{1}{2}\left(\mathcal{H}[\Omega]\right)^2 - \frac{\Omega^2}{2}.$$

Then use this equation and equation $(*)$ to derive the equation

$$\frac{\partial F}{\partial \tau} = -\frac{i}{2}F^2$$

for the function $F = \Omega + i\mathcal{H}[\Omega]$.

Hint: Use the identities $\mathcal{H}[\mathcal{H}[f]] = -f$ and
$\mathcal{H}[fg] = f\mathcal{H}[g] + g\mathcal{H}[f] + \mathcal{H}[\mathcal{H}[f]\mathcal{H}[g]]$. ***(7 marks)***

(iii) You are given that ω satisfies the initial condition $\omega = \sin x$ at $t = 0$.

(a) Find the initial condition for F at $\tau = 0$. Then solve the equation for F subject to this initial condition. Use this solution to obtain the expression for ω.

Hint: Use the formula $\mathcal{H}[\sin x] = -\cos x$. ***(10 marks)***

(b) Find the restriction on γ when the solution for ω does not break down. ***(4 marks)***

4 Consider a two-dimensional potential flow of an incompressible fluid, $\nabla \cdot \boldsymbol{v} = 0$, $\nabla \times \boldsymbol{v} = 0$, where $\boldsymbol{v} = (u, v)$ is the velocity. The velocity components are expressed in terms of the flux function Ψ as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}.$$

(i) Show that the complex potential $W = \Phi + i\Psi$, where Φ is the velocity potential, is an analytic function of the complex variable $z = x + iy$. Express $\dfrac{dW}{dz}$ in terms of u and v. ***(4 marks)***

(ii) Show that the velocity circulation around a simple (i.e. without self-intersections) smooth closed contour C is equal to the real part of the variation of the complex potential around this contour, that is

$$\oint_C (u\,dx + v\,dy) = \Re(\Delta W).$$

(4 marks)

(iii) The complex potential of a vortex street is given by

$$W = \frac{\kappa}{2\pi i} \log \sin \frac{\pi z}{\ell},$$

where κ and ℓ are positive constants.

(a) A simple smooth closed contour C encloses the singularity of the complex potential W at $z = j\ell$, and it does not enclose any other singularity. Show that the velocity circulation around this contour is equal to κ.

Hint: You can use without proof that the variation of $\log f(z)$ around the contour C is zero, where $f(z)$ is an analytic function satisfying the conditions that $f(z) \neq 0$ and $f(z)$ is regular in the domain enclosed by C. ***(6 marks)***

(b) You are given that the complex velocity potential for a single vortex situated at $z = 0$ is $\dfrac{\kappa}{2\pi i} \log \dfrac{\pi z}{\ell}$. Calculate the complex velocity $u_0 - iv_0$ of this vortex eliminating its "self-effect", that is using the reduced complex potential

$$W' = \frac{\kappa}{2\pi i} \left(\log \sin \frac{\pi z}{\ell} - \log \frac{\pi z}{\ell} \right),$$

and thus show that this vortex is at rest. ***(3 marks)***

 Question 4 continued on next page

4 (continued)

(c) Now assume that the vortex at $z = 0$ is displaced by a small distance z_0. Calculate the complex velocity $u_0 - iv_0$ of the displaced vortex only retaining the term proportional to z_0. Use this result to show that the vortex street is unstable.

Hint: Use the relations $u_0 = \dfrac{dx_0}{dt}$ and $v_0 = \dfrac{dy_0}{dt}$, where (x_0, y_0) are the vortex coordinates. ***(8 marks)***

5 A smooth simple (i.e. without self-intersections) curve C is defined by equations

$$x = x_0(\alpha), \quad y = y_0(\alpha),$$

where $\alpha \in R^1$ is the arclength of C, meaning, in particular, that

$$\left(\frac{dx_0}{d\alpha}\right)^2 + \left(\frac{dy_0}{d\alpha}\right)^2 = 1.$$

Curvilinear coordinates in the vicinity of C are α and s, where s is measured along the normal direction to C, and (α, s) is a right-oriented coordinate system.

(i) Show that the transformation from coordinates (α, s) to Cartesian coordinates is given by

$$x = x_0(\alpha) - s\frac{dy_0}{d\alpha}, \quad y = y_0(\alpha) + s\frac{dx_0}{d\alpha}.$$

(4 marks)

(ii) Consider a planar motion of ideal homogeneous incompressible fluid. You are given that C is a fluid contour, (α, s) are Lagrangian coordinates, meaning that the fluid motion is described by the equations

$$x = x(t, \alpha, s), \quad y = y(t, \alpha, s).$$

Show by the direct calculation of the time derivative that the Jacobian of transformation from (α, s) to (x, y) is independent of time, and then calculate $J(\alpha, s)$. ***(5 marks)***

(iii) You are given that the velocity components are expressed in terms of the flux function Ψ as

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}.$$

You are also given that the vorticity is defined by $\omega = \gamma(\alpha)\delta(s)$, where $\gamma(\alpha) \to 0$ as $|\alpha| \to \infty$.

(a) Show that Ψ satisfies the Poisson's equation

$$\nabla^2 \Psi = -\omega.$$

Use the expression for the general solution to this equation,

$$\Psi(\boldsymbol{x}) = \frac{1}{2\pi}\int_{R^2} \omega(\boldsymbol{x}') \log \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}',$$

to show that, for the particular form of ω defined above, Ψ is given by

$$\Psi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma(\beta) \log \frac{1}{|z(\alpha, s) - z(\beta)|} d\beta.$$

where $z = x + iy$. ***(5 marks)***

 Question 5 continued on next page

5 (continued)

(b) Use the expression for Ψ to derive the Birkhoff-Rott equation

$$\frac{\partial z^*}{\partial t} = -\frac{i}{2\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\gamma(\beta)\, d\beta}{z(\alpha) - z(\beta)}, \qquad (\dagger)$$

where $\mathcal{P}$ indicates the principal Cauchy part of integral and the asterisk indicates complex conjugate. Explain why we need to use the principal Cauchy part of integral in this equation. Also explain why this equation remains valid even if we relax the assumption that $\gamma(\alpha) \to 0$ as $|\alpha| \to \infty$ and assume instead that $\gamma(\alpha) \to \text{const} \neq 0$ as $|\alpha| \to \infty$. ***(8 marks)***

(c) You are now given that $\gamma(\alpha) = \text{const}$. Show that, in this case, the Birkhoff-Rott equation ($\dagger$) admits a solution describing the straight vortex sheet, $z = \alpha$.

Use the definition of the principal Cauchy part of integral in terms of limits. ***(3 marks)***

End of Question Paper