



SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester
2015-16

Fields

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

- 1 (i) For each of the subsets J_1, J_2 of \mathbb{C} specified below determine, with justification, whether it is a subfield of \mathbb{C} (where $i^2 = -1$):
- (a) $J_1 = \{a + bi\sqrt{11} : a, b \in \mathbb{Q}\}$, (4 marks)
- (b) $J_2 = \{a + bi + ci\sqrt{-4} : a, b, c \in \mathbb{Q}\}$. (3 marks)
- (ii) Let $L = \mathbb{Q}(\sqrt{5}, i)$ and $\alpha = \sqrt{5} + i$.
- (a) Find $[L : \mathbb{Q}]$. Justify your response. (5 marks)
- (b) Show that $L = \mathbb{Q}(\alpha)$. (5 marks)
- (c) Express the element $\frac{1}{1 + \sqrt{5} + i}$ in the form $\alpha + \beta\sqrt{5} + \gamma i + \delta i\sqrt{5}$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$. (5 marks)
- (iii) State Eisenstein's Irreducibility Criterion. (3 marks)
- 2 (i) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. (2 marks)
- (ii) Give a proof of the degrees formula. (9 marks)
- (iii) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{2}i)$ and $b = \sqrt{2}i$.
- (a) Find a \mathbb{Q} -basis of the field L . (5 marks)
- (b) Find the minimal polynomial of the element b over the field \mathbb{Q} . (5 marks)
- (c) Is $L = \mathbb{Q}(b)$? Justify your response. (4 marks)

- 3 (i) Let L and M be subfields of the fields of complex numbers \mathbb{C} such that $n = [L : \mathbb{Q}]$ and $m = [M : \mathbb{Q}]$ are co-prime natural numbers (i.e. the greatest common divisor of n and m is 1). Let LM be the subfield of \mathbb{C} generated by the fields L and M . What is $[LM : \mathbb{Q}]$? Justify your response. *(7 marks)*
- (ii) Define the content of a polynomial and find the content of the product of polynomials $p = p_1 p_2 \cdots p_{10}$ where $p_i = \sum_{j=i}^{10} 2^j x^n$. *(5 marks)*
- (iii) Let $f, g \in \mathbb{Z}[x]$ be nonzero polynomials with integer coefficients. What is the content $c(fg)$ of their product? Justify your response (there is no need to prove Gauss's Lemma). *(6 marks)*
- (iv) Let $f \in \mathbb{Z}[x]$ be a primitive polynomial. Prove that if the polynomial f is reducible in $\mathbb{Q}[x]$ then $f = pq$ for some non-constant polynomials $p, q \in \mathbb{Z}[x]$. *(7 marks)*
- 4 Let $K \subseteq L$ be subfields of \mathbb{C} .
- (i) Define the group $G(L/K)$. Is it true or not that if $L = K(\alpha)$ where α is an algebraic number over K then the group $G(L/K)$ is a finite group (no justification is required)? *(4 marks)*
- (ii) Find the group $G(\mathbb{Q}(\sqrt{6})/\mathbb{Q})$. *(8 marks)*
- (iii) Let p be a prime number, \mathbb{F}_{p^n} and \mathbb{F}_{p^m} be finite fields that contain p^n and p^m elements, respectively.
- (a) Prove that $\mathbb{F}_{p^n} \subseteq \mathbb{F}_{p^m}$ if and only if $n|m$. *(8 marks)*
- (b) Let n be a natural number such that $n \geq 2$. Is it true or not that $\cup_{i=1}^{\infty} \mathbb{F}_{p^{in}} = \overline{\mathbb{F}_p}$? Justify your response ($\overline{\mathbb{F}_p}$ is an algebraic closure of the field \mathbb{F}_p). *(5 marks)*

End of Question Paper