The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2015-2016

Analytical Dynamics and Classical Field Theory — **3 Hours**

Answer five questions. If you answer more than five questions, only your best five will be counted.

1 Consider a system with N degrees of freedom. The Lagrange–function is given by $L = L(q_i, \dot{q}_i, t)$, with $i = 1, ..., N$.

(i) Write down the Euler–Lagrange equation corresponding to the coordinate q_i. Furthermore, by explicit calculation, show that

$$\frac{d}{dt}\left(\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = -\frac{\partial L}{\partial t}.$$

Hence, show that if L does not explicitly depend on time, the quantity in the brackets is conserved. What is the interpretation of the quantity? ***(7 marks)***

(ii) The Lagrange–function for a particle of mass m and charge e is given by

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - e\Phi + e\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right),$$

where x, y, z are the coordinates of the particle, Φ is the scalar potential, A_x, A_y, A_z are the components of the vector potential $\mathbf{A} = (A_x, A_y, A_z)$ and $\dot{x} = dx/dt$, etc. Show that the Euler–Lagrange equations lead to the equation

$$m\ddot{\mathbf{x}} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right),$$

where $\mathbf{x} = (x, y, z)$ is the position vector of the particle, $\mathbf{v} = d\mathbf{x}/dt$ is the velocity, $\mathbf{E} = -\nabla\Phi - \dfrac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Hint: Since x, y, and z appear in L on the same footing, it is sufficient to consider one component only (e.g. the x–coordinate). ***(13 marks)***

2 Consider a system with N degrees of freedom, which is specified by N generalised coordinates q_i $(i = 1, ..., N)$ and a Lagrangian $L = L(q_i, \dot{q}_i, t)$.

(i) State Hamilton's principle. ***(2 marks)***

(ii) Define the canonical momenta P_i and define the Hamiltonian $H = H(q_i, P_i, t)$. ***(3 marks)***

(iii) Consider the action

$$S = \int_{t_1}^{t_2} \left[\sum_{i=1}^{N} P_i \dot{q}_i - H \right] dt.$$

Show that extremising this action with respect to q_i and P_i leads to Hamilton's equations. Hint: You may want to use the Euler–Lagrange equations for q_i and P_i. ***(8 marks)***

(iv) A particle moves along the x-axis under the influence of the potential $V = \frac{1}{2}kx^2 + \lambda x^4$. Write down the Hamilton–function $H(x, P)$ for this system but do not assume that H is the sum of the kinetic and potential energies. Find the equation of motion for the particle from Hamilton's equations. Is H conserved? ***(7 marks)***

3 (i) Consider two functions $f = f(q_i, P_i)$ and $g = g(q_i, P_i)$, where $i = 1, ..., N$ and N is the number of degrees of freedom. Define the Poisson bracket $\{f, g\}$. ***(2 marks)***

For the rest of this question, we consider a system which is described by one degree of freedom. The generalised coordinate is denoted by q and the canonical momentum by P.

(ii) Consider a coordinate transformation $q \to Q = Q(q, P), P \to \tilde{P} = \tilde{P}(q, P)$ and denote the Poisson brackets in the two coordinate systems with

$$\{f, g\}_{q,P} \equiv \frac{\partial f}{\partial q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial q}$$
$$\{f, g\}_{Q,\tilde{P}} \equiv \frac{\partial f}{\partial Q}\frac{\partial g}{\partial \tilde{P}} - \frac{\partial f}{\partial \tilde{P}}\frac{\partial g}{\partial Q}.$$

Show that $\{f, g\}_{q,P} = \{f, g\}_{Q,\tilde{P}}$ only if $\{Q, \tilde{P}\}_{q,P} = 1$. Hint: It is useful to first show that $\{f, g\}_{q,P} = \{f, g\}_{Q,\tilde{P}} \cdot \{Q, \tilde{P}\}_{q,P}$. ***(13 marks)***

(iii) Show that $\{Q, \tilde{P}\}_{q,P} = 1$ for the following coordinate transformation: ***(5 marks)***

$$q \to Q = \log\left(\frac{\sinh q}{P}\right) , \qquad P \to \tilde{P} = P\frac{\cosh q}{\sinh q}$$

4 The electromagnetic field strength tensor is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is a four–vector.

(i) How does this tensor transform under general Lorentz transformations?

(3 marks)

(ii) In general, $F_{\mu\nu}$ can be written in terms of the electric field $\mathbf{E} = (E_x, E_y, E_z)$ and magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ as follows

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} .$$

Assume that in an inertial frame, called frame R_1, $F_{\mu\nu}$ has the form above. In this system, the electric field is non–zero, i.e. $\mathbf{E} = (E_x, E_y, E_z)$ is not vanishing but the magnetic field vanishes, i.e. $\mathbf{B} = (B_x, B_y, B_z) = (0, 0, 0)$. Consider now a second inertial frame, called R_2, moving with velocity $\mathbf{v} = (v, 0, 0)$ in the x–direction relative to frame R_1. The Lorentz–transformation between frames R_1 and R_2 is given by the matrix

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Assume that in frame R_2 the electromagnetic field strength tensor has the same form as in frame R_1, i.e. it can be written as

$$F'_{\mu\nu} = \begin{pmatrix} 0 & E'_x & E'_y & E'_z \\ -E'_x & 0 & -B'_z & B'_y \\ -E'_y & B'_z & 0 & -B'_x \\ -E'_z & -B'_y & B'_x & 0 \end{pmatrix} .$$

Determine the magnetic field $\mathbf{B}'$ in the system R_2 and show that it doesn't vanish (i.e. $\mathbf{B}' = (B'_x, B'_y, B'_z) \neq \mathbf{0}$).

(17 marks)

5 For this question, you can use the expression for $F_{\mu\nu}$ given in Question 4.

(i) Show that $F_{\mu\nu}F^{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right)$, where $\mathbf{E} = (E_x, E_y, E_z)$ is the electric field and $\mathbf{B} = (B_x, B_y, B_z)$ is the magnetic field. Is this an invariant under a Lorentz–transformation? ***(8 marks)***

(ii) Consider the following Lagrangian, describing the interaction between the electromagnetic field and a real massless scalar field ϕ:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - e^{-\phi}F_{\mu\nu}F^{\mu\nu}.$$

Use the Euler–Lagrange equation for ϕ to show that the equation of motion for the scalar field is given by $\partial^\mu\partial_\mu\phi - 2e^{-\phi}(\mathbf{B}^2 - \mathbf{E}^2) = 0$. Add a mass term to the Lagrangian above. What is the equation of motion for the scalar field then? ***(12 marks)***

6 Consider the following two Lagrangians $\mathcal{L}_c$ and $\mathcal{L}_s$ for a complex scalar field ϕ and two real scalar fields ϕ_1 and ϕ_2:

$$\mathcal{L}_c = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\bar{\phi}\right) - \frac{m^2}{2}\phi\bar{\phi} \qquad (1)$$

$$\mathcal{L}_s = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_1\right)\left(\partial_\nu\phi_1\right) - \frac{m^2}{2}\phi_1^2 + \frac{1}{2}\eta^{\mu\nu}\left(\partial_\mu\phi_2\right)\left(\partial_\nu\phi_2\right) - \frac{m^2}{2}\phi_2^2 \qquad (2)$$

In these equations, m is a constant and $\bar{\phi}$ is the complex conjugate of ϕ.

(i) By setting $\phi = \phi_1 + i\phi_2$, show that the two Lagrangians $\mathcal{L}_s$ and $\mathcal{L}_c$ are equivalent. ***(7 marks)***

(ii) Use the Euler–Lagrange equations for ϕ_1 and ϕ_2 to find the equations of motion for the two fields ϕ_1 and ϕ_2. Using the equations of motion for ϕ_1 and ϕ_2, find the equation of motion for $\phi = \phi_1 + i\phi_2$. ***(5 marks)***

(iii) Show that the action $S = \int \mathcal{L}_c d^4x$ is invariant under the transformation

$$\phi \to \phi' = e^{i\alpha}\phi, \qquad x^\mu \to x'^\mu = x^\mu \qquad (3)$$

where α is a real constant. Find the (conserved) Noether current. You are given that the Noether current can be written as

$$j^\mu = -\frac{\partial\mathcal{L}_c}{\partial(\partial_\mu\phi)}F_1(\phi, \partial\phi) - \frac{\partial\mathcal{L}_c}{\partial(\partial_\mu\bar{\phi})}F_2(\phi, \partial\bar{\phi}),$$

for general infinitesimal transformations of the form (ϵ is a small parameter)

$$\begin{aligned} x'^\mu &= x^\mu \qquad (4) \\ \phi'(x') &= \phi(x) + \epsilon F_1(\phi, \partial\phi) \\ \bar{\phi}'(x') &= \bar{\phi}(x) + \epsilon F_2(\phi, \partial\phi) \end{aligned}$$

Hint: Assume that α is small and expand the transformations (3) to bring them into the form (4). ***(8 marks)***

End of Question Paper