The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2007–8

PAS201 Statistics Core — 2 hours

Marks will be awarded for your best ***three*** *answers.*

RESTRICTED OPEN BOOK EXAMINATION

Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.

There are 99 marks available on the paper.

1 Suppose that the random variable X follows the beta distribution $X \sim Be(\alpha, \beta)$, with p.d.f.

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1; \quad \alpha, \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that if $\alpha = 1$, $f(x)$ is reduced to $f(x) = \beta(1-x)^{\beta-1}$, for $0 < x < 1$.

(6 marks)

(ii) Show that the k-th moment of X is

$$E(X^k) = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{(\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+k-1)}.$$

(12 marks)

(iii) If $\alpha = 2$ and $\beta = 3$ find the coefficient of skewness of X. Hence determine whether X is symmetric, negatively, or positively skewed random variable.

(8 marks)

(iv) Define the random variable $Y = X^{1/3}$. Find the p.d.f. of Y.

(7 marks)

2 Three random variables have joint p.d.f.

$$f_{XYZ}(x,y,z) = \begin{cases} \dfrac{c}{xy}\exp(-yz), & 1 < x, y < 2, \quad z > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that the constant c is $c = \dfrac{2}{\log 2}$. ***(6 marks)***

(ii) Find the marginal joint p.d.f. of $(X,Y)^T$, and the marginal p.d.f.'s of X and Y. Show that X and Y are independent, but X, Y and Z are not independent. ***(10 marks)***

(iii) Calculate the mean and variance of X and Y. Also calculate the mean vector of $\mathbf{W} = (X,Y)^T$ and the covariance matrix of $\mathbf{W}$. ***(9 marks)***

(iv) For $1 < x, y < 2$, find the joint distribution function $F_{XY}(x,y)$ of $(X,Y)^T$ and calculate the probability $P(X \leq 3/2, Y \leq 4/3)$. ***(8 marks)***

3 (i) Suppose that the random vector $\mathbf{X} = (X_1, X_2, X_3)^T$ follows a multivariate normal distribution, where X_1, X_2, X_3 have zero mean and variances 1,2,3 respectively. Assume that X_1, X_2 are independent, X_1, X_3 are independent and that the correlation coefficient of X_2, X_3 is $1/2$. Define the random variables Y_1 and Y_2 as $Y_1 = 2X_1 - X_3$ and $Y_2 = X_2 + X_3$. Find the distribution of $(Y_1, Y_2)^T$ and the correlation coefficient of Y_1, Y_2. ***(8 marks)***

(ii) Consider two independent random variables X and Y. It is assumed that X follows the exponential distribution with p.d.f. $f_X(x) = 3\exp(-3x)$, for $x \geq 0$ and Y follows the gamma distribution with p.d.f. $f_Y(y) = \dfrac{2^3}{\Gamma(3)} y^2 \exp(-2y)$, for $y \geq 0$, where $\Gamma(.)$ denotes the gamma function. Find the p.d.f. of $\dfrac{X}{Y}$, and show that $\dfrac{X}{Y}$ is not independent of Y. ***(25 marks)***

4 (i) A random sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population of the Laplace distribution so that

$$f(x_i|\theta) = \frac{1}{2\theta} \exp\left(-\frac{|x_i|}{\theta}\right), \quad -\infty < x_i < +\infty, \quad \theta > 0.$$

Find the maximum likelihood estimate of θ, based on that sample.

(17 marks)

(ii) A random sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population of the Rayleigh distribution so that

$$f(x_i|s) = \frac{x_i}{s^2} \exp\left(-\frac{x_i^2}{2s^2}\right), \quad x_i \geq 0, \quad s \neq 0.$$

Find the maximum likelihood estimate of $\theta = s^2$, based on that sample.

(16 marks)

End of Question Paper