Data provided: Graph paper, Laplace Transform table

AMA312

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2007–8

Classical Control Theory — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 The transfer functions of two linear systems are

$$G_1(s) = \frac{V(s)}{U(s)} = \frac{s+1}{s^2+2s+5} \quad \text{and} \quad G_2(s) = \frac{Y(s)}{V(s)} = \frac{s}{s^2+5s+6} \; .$$

(i) Give the two linear differential equations relating the variables $u(t)$, $v(t)$; and $v(t)$, $y(t)$. Give the impulse responses of the systems G_1 and G_2.

(8 marks)

(ii) For the series connection $G = G_1G_2$ compute the overall transfer function $\dfrac{Y(s)}{U(s)}$. Determine the OL poles and zeros, and establish whether the system is open-loop stable.

(3 marks)

(iii) Now close the loop to obtain a constant-gain negative feedback system. Calculate the closed-loop transfer function $H(s)$ and the characteristic equation for the closed-loop poles.

Use the Routh-Hurwitz criterion to show that the closed-loop system is stable for all gain values $k \geq 0$. ***(7 marks)***

(iv) Hence, sketch the root locus plot; computing the relevant angles of departure and using the mid-point approximation for a breakaway point.

(7 marks)

2 (i) Consider the unity negative feedback system with open-loop transfer function

$$G(s) = \frac{1}{s(s+2)}.$$

Compute the closed-loop transfer function $H(s)$ and the error transfer function, $E(s)/R(s)$, where $r(t)$ is the reference input and $e(t) = r(t) - y(t)$ is the error.

(5 marks)

(ii) Using the Partial Fractions Expansion, compute the error $e(t)$, $t \geq 0$, for the cases where the reference input is:

(a) a unit step, $r(t) = h(t)$ and (b) a unit ramp $r(t) = t\ h(t)$.

Show that the steady-state error, $e_{ss} = \lim_{t\to\infty} e(t)$, is zero for the unit step, but is equal to a constant for a unit ramp. Calculate this constant.

(11 marks)

(iii) Now explore the claim that the modified open-loop transfer

$$\tilde{G}(s) = \frac{s+1}{s^2(s+2)}$$

can track both steps and ramps with *zero steady-state error*.

First check, using the Routh–Hurwitz criterion, that the closed-loop transfer function $E(s)/R(s)$ is stable.

Next, appealing to the *form* of the partial fractions expansion (*without computing all the coefficients*), show that the steady-state error to both a unit step and a unit ramp is zero.

Finally, show that the steady-state error to the *parabolic* input

$$r(t) = \frac{t^2}{2}\ h(t)$$

is a constant. Calculate this constant.

(9 marks)

3 (i) Sketch the Nyquist plot of the transfer function

$$G(s) = \frac{s+10}{(s+1)(s+2)(s+3)},$$

determining all frequencies at crossings of the *real* axis. ***(9 marks)***

(ii) This linear system $G(s)$ is placed in a constant-gain negative feedback configuration.

 Question 3 continued on next page

3 (continued)

Sketch the root locus plot, specifying all the crossover points on the imaginary axis. These should be computed from the Routh–Hurwitz table. You can use the mid-point approximation for any breakaway points.

Confirm that the critical value of the gain, k_c, and the critical frequencies, $\pm\omega_c$, at crossover, are the same as those obtained from the Nyquist plot and the application of the Nyquist stability criterion.

(12 marks)

(iii) If, instead of the given $G(s)$, we had

$$\tilde{G}(s) = \frac{s+d}{(s+1)(s+2)(s+3)},$$

where d is some parameter, use the centre of asymptotes formula to argue that the closed-loop system is stable for all gain values $k \geq 0$, provided $d < 6$.

(4 marks)

4 (i) Draw the root locus plot for the constant-gain negative feedback system with OL transfer function

$$G(s) = \frac{s-1}{s(s+2)((s+1)^2+4)}$$

and hence conclude that the closed-loop system is unstable for all values of the gain $k > 0$.

(11 marks)

(ii) We aim to modify the above system so that it is stable for *some* values of k. Let us introduce a *second unstable* open-loop zero:

$$\tilde{G}(s) = \frac{(s-1)(s-0.5)}{s(s+2)((s+1)^2+4)}.$$

Using the Routh–Hurwitz table, decide whether this strategy is successful and, if it is, give the range of gain values for which we get closed-loop stability.

(11 marks)

(iii) Give a quick argument (using the root locus plot, for example), to show that the system with an extra *stable OL pole*,

$$\bar{G}(s) = \frac{s-1}{s(s+1)(s+2)((s+1)^2+4)},$$

will not work (i.e., the CL system will be unstable for all $k \geq 0$.)

(3 marks)

5 (i) You are given the linear differential equation

$$2\ddot{y} + 8\dot{y} + 6y = 3\dot{u} + u \ .$$

Solve for the output $y(t)$, $t \geq 0$, when $y(0) = 0$, $\dot{y}(0) = 1$ and the input

$$u(t) = \cos(2t)\ h(t) \ .$$

(10 marks)

(ii) Sketch the Nyquist plot for the constant-gain negative feedback system, with OL transfer function

$$G(s) = \frac{(s+8)}{(s^3 + 2s^2 + 5s + 8)} \ .$$

Check that the closed-loop system with gain $k_0 = 0.25$ is stable and find the gain and phase margins for this nominal gain value k_0. [The equation you will find when computing the phase margin frequency is cubic in ω^2 and should be:

$$16\omega^6 - 96\omega^4 - 113\omega^2 + 960 = 0 \ ;$$

you do not need to solve this equation, the real root you need is $\omega^2 \simeq 5.0377$.]

(12 marks)

(iii) Suppose now that the actual transfer function has some unmodelled high–frequency dynamics, so that the transfer function is

$$\tilde{G}(s) = \frac{10(s+8)}{(s+10)(s^3 + 2s^2 + 5s + 8)} \ .$$

Decide whether the additional phase contribution makes the system unstable.

(3 marks)

Table of Laplace Transform Pairs

Time Function	Laplace Transform
$h(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\frac{t^n e^{-at}}{n!}$	$\frac{1}{(s+a)^{n+1}}$
$\cos \omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
$\sin \omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

End of Question Paper