The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2007-2008**

MATHEMATICS III (Control and Aerospace Engineering) **2 hours**

Answer **four** *questions. You are advised* **not** *to answer more than four questions; if you do, only your best four will be counted.*

1. In this problem the following matrix A will be used.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

a) Reduce the matrix A to echelon form and show that it has rank equal to 2. ***(5 marks)***

b) Find the null space and the range space of the matrix A. What is the dimension of each of the two spaces?

(8 marks)

c) Determine the rank of the augmented matrix $(A|b)$ of the following matrix equations. Do the equations have a solution? If so, determine the full solution, including all the free parameters.

$$A\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 4 \end{pmatrix}, \qquad A\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 5 \end{pmatrix}$$

(12 marks)

2. a) A and B are $n \times n$ matrices, and it is given that B has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $X_1, X_2, \ldots, X_n$. If $B = T^{-1}AT$, show that A has the same eigenvalues with corresponding eigenvectors $TX_1, TX_2, \ldots, TX_n$. ***(4 marks)***

b) Show that the matrix $A = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ has eigenvalues $\pm\sqrt{5}$. Find the corresponding *normalised* eigenvectors. ***(12 marks)***

c) Hence, or otherwise, construct the matrix T so that $B = T^{-1}AT$ is a diagonal matrix. ***(3 marks)***

d) What are the eigenvalues of the matrices A^2 and $A^3 - A$? Find the matrix $T^{-1}A^2T$. ***(6 marks)***

3. a) For the 2×2 matrix A, it is given that

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

and that A satisfies its own characteristic equation (the Cayley-Hamilton theorem) of the form

$$A^2 = aA + bI$$

where a and b are constants.

Hence deduce that

$$e^A = \alpha A + \beta I$$

for some constants α and β. ***(10 marks)***

b) Show that the matrix $B = \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix}$ has eigenvalues 3 and -2. ***(2 marks)***

Hence, evaluate the matrices B^6 and e^B. ***(13 marks)***

[*Note*: It may be assumed that all the infinite series that occur in this question converge.]

4. a) Determine whether the sets of vectors

(i) $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 10 \\ -3 \\ 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 2 \end{pmatrix}$

(8 marks)

are linearly dependent or linearly independent.

b) Show that the equations

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ -3 \\ 2 \end{pmatrix}$$

have a unique solution. ***(5 marks)***

c) Show that the equations

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 2 \end{pmatrix}$$

have no solution. ***(4 marks)***

d) Given that the nearest least squares solution, u^*, of the matrix equation $Au = b$ is

$$u^* = (A^T A)^{-1} A^T b,$$

determine the nearest least squares solution of the set of equations in part (c).

(8 marks)

5. a) Show that the quadratic form

$$x_1^2 + x_2^2 + 3x_1x_2$$

can be written in the form X^TAX, where X is the vector of the variables and A is a 2×2 matrix.

(4 marks)

Find the matrix U so that $X = UY$, with $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, transforms the quadratic form to

$$\lambda y_1^2 + \mu y_2^2.$$

(10 marks)

Is the quadratic form positive definite, negative definite, or indefinite?

(1 marks)

b) Find the stationary points of the function

$$f(x,y) = x^3 + y^3 + 3xy$$

and investigate their nature.

(10 marks)

6. a) Find the matrix representation of the map

$$T = 2\frac{d^2}{dx^2} - \frac{d}{dx} + 1$$

acting on quadratics.

(8 marks)

Hence, find a solution of the equation $Tu = x^2$.

(5 marks)

b) Use the method of Lagrange multipliers to determine the minimum of

$$f(x,y) = x^2 + y^2$$

subject to the constraint

$$\phi(x,y) = 2x^2 + y^2 - 1 = 0\,.$$

(12 marks)

End Of Question Paper