The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2008-2009**

Mathematics III (Control) **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1. (i) Consider the three sets of linear equations:

(a)
$$\begin{aligned} x + 3y - z &= -5 \\ x + 4y \phantom{{}-z} &= 2 \\ 2x + 6y - z &= -1 \end{aligned}$$

(b)
$$\begin{aligned} x + 3y - z &= -5 \\ x + 4y \phantom{{}-z} &= 2 \\ 2x + 7y - z &= -1 \end{aligned}$$

(c)
$$\begin{aligned} x + 3y - z &= -5 \\ x + 4y \phantom{{}-z} &= 2 \\ 2x + 7y - z &= -3 \end{aligned}$$

Show that one set has a unique solution, one has many solutions and one has no solutions.

(20 marks)

(ii) For the set which has many solutions, give them in a parametric form.

(5 marks)

2. (i) State, giving reasons, which of the following functions are linear transformations:

$$T_1(x, y, z) = (3x - 2y + 4z,\ x + y - 2z,\ 2x - yz)$$

$$T_2(x, y, z) = (x + 3y - z,\ 2x - y + 5z,\ x + 10y - 8z)$$

$$T_3(x, y, z) = (2x + y - 5z,\ x + 2y + z,\ 3x + 4y - 2z)$$

(10 marks)

(ii) Show that one of the above linear transformations is non-singular and find its inverse.

(10 marks)

(iii) Show that another is singular and find a basis for its kernel.

(5 marks)

3. Let the linear operator T be represented by the matrix

$$A = \begin{pmatrix} 1 & 5 & -3 \\ -4 & 13 & -6 \\ -4 & 11 & -4 \end{pmatrix}.$$

(i) Show that $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is an eigenvector of A and find the associated eigenvalue.

(4 marks)

(ii) Show that 2 is an eigenvalue of A and find an associated eigenvector.

(8 marks)

(iii) Find the trace of A and hence or otherwise find a third eigenvalue of A and an associated eigenvector.

(6 marks)

(iv) Write down the characteristic polynomial of A and use the Cayley-Hamilton theorem to evaluate $A^3 - 10A^2 + 30A$.

(7 marks)

4. Let A be the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{pmatrix}.$$

(i) Find the eigenvalues of A and state the algebraic multiplicity of each.

(5 marks)

(ii) Find the eigenspace associated with each eigenvalue and hence state the geometric multiplicity of each eigenvalue.

(6 marks)

(iii) Write down the Jordan form J of A and show that $J = P^{-1}AP$, where

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 4 & 4 \end{pmatrix}$$

(6 marks)

(iv) Hence or otherwise find the general solution of the differential equation

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = 3\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 4x.$$

(8 marks)

5. (i) Give a matrix representation of the quadratic form:

$$Q(x, y, z) = 3x^2 + 2y^2 + z^2 - 4xy + 4yz.$$

(4 marks)

(ii) Find the eigenvalues and normalised eigenvectors for the matrix obtained in part (i).

(16 marks)

(iii) Write $Q(x, y, z)$ as a sum or difference of squares and state whether it is positive definite, negative definite or indefinite.

(3 marks)

(iv) Give a brief description of the surface $Q(x, y, z) = 10$.

(2 marks)

6. (a) Find the stationary points of the function

$$z = f(x, y) = x^4 - 4x^3 + 4x^2 - 3y^2 + 6y$$

and investigate their nature.

(14 marks)

(b) Use Lagrange multipliers to find the maximum value of the function $f(x, y) = 60x + 40y$ on the ellipse $4x^2 + y^2 = 25$.

(11 marks)

End of Question Paper