The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2008–9

PAS201 Statistics Core — 2 hours

Marks will be awarded for your best ***three*** *answers.*

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 99 marks available on the paper.

1 Suppose that the random variable X follows the exponential distribution $X \sim Exp(\lambda)$, with p.d.f.

$$f(x) = \begin{cases} \lambda \exp(-\lambda x), & x > 0; \quad \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the k-th moment of X, i.e. $E(X^k)$, for some positive integer k. ***(7 marks)***

(ii) Find the distribution function of X and calculate the probability $P(X \leq 1)$. ***(8 marks)***

(iii) Calculate the coefficient of skewness of X and comment on the skewness of X. ***(10 marks)***

(iv) Define the random variable $Y = X^{1/3}$. Find the p.d.f. of Y. ***(8 marks)***

2 Three random variables have joint p.d.f.

$$f_{XYZ}(x,y,z) = \begin{cases} cx^2(y-1)z^3, & 1 < x, z < 1, \quad 1 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that the constant c is $c = 24$. ***(6 marks)***

(ii) Calculate the marginal p.d.f. of X, the marginal p.d.f. of Y and the marginal p.d.f. of Z, and show that X, Y, Z are independent. ***(10 marks)***

(iii) Find the joint marginal p.d.f. of X, Y and show that $(X, Y)^T$ is independent of Z. Calculate the mean vector of $(X, Y)^T$. ***(9 marks)***

(iv) For $0 < x, z < 1$ and $1 < y < 2$, find the joint distribution function of X, Y, Z and calculate the probability $P(X \leq 1/3, Y \leq 3/2, Z \leq 1/2)$. ***(8 marks)***

3 (i) Suppose that the random vector $\mathbf{X} = (X_1, X_2, X_3)^T$ follows a multivariate (trivariate) normal distribution, where X_1, X_2, X_3 have means 2, 2, 1 and variances 1, 1, 1 respectively. Assume that X_1, X_3 are uncorrelated, the correlation coefficient of X_1, X_2 is 1/2 and the correlation coefficient of X_2, X_3 is 2/3.

(a) Write down the mean vector, the covariance matrix and the distribution of the random vector $\mathbf{X}$. ***(8 marks)***

(b) Find the mean vector, the covariance matrix and the distribution of the random vector

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Calculate the correlation coefficient of Y_1 and Y_2. ***(11 marks)***

(ii) Consider two independent random variables X and Y. It is assumed that X follows the gamma distribution $X \sim Ga(3, 2)$ and Y follows the gamma distribution $Y \sim Ga(2, 1)$ with p.d.f.'s

$$f_X(x) = \frac{3^2}{\Gamma(3)} x^{3-1} \exp(-2x) \quad \text{and} \quad f_Y(y) = \frac{1^2}{\Gamma(2)} y^{2-1} \exp(-y),$$

for $x, y > 0$, where $\Gamma(\alpha)$ denotes the gamma function with argument α. Find the p.d.f. of the random variable Y/X. [HINT: Define two new random variables $U = Y/X$ and $V = X$]. ***(14 marks)***

4 (i) A random sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population of the beta distribution $X_i|\theta \sim B(\theta, 1)$, so that

$$f(x_i|\theta) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x_i^{\theta-1}(1-x_i)^{1-1},$$

where $\Gamma(\theta)$ denotes the gamma function with argument θ. Find the maximum likelihood estimate of θ, based on that sample. ***(17 marks)***

(ii) A random sample $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a population of the distribution with p.d.f.

$$f(x_i|\theta) = 2\theta x_i \exp\left(-\theta x_i^2\right), \quad x_i \geq 0, \quad \theta > 0.$$

Find the maximum likelihood estimate of θ, based on that sample.
(16 marks)

End of Question Paper