The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2008–2009**

Vector spaces and Fourier theory **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) For each of the following subsets of $\mathbb{R}^3$, say whether they are a subspace or not, justifying your answers:

(a) $U_1 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x = 1\}$;

(b) $U_2 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x^2 = y^2\}$;

(c) $U_3 = \{(x, y, z)^T \in \mathbb{R}^3 \mid 2x = y + z\}$. ***(6 marks)***

(ii) If U and W are subspaces of a vector space V, define their sum $U + W$. ***(2 marks)***

(iii) In the vector space $V = \mathbb{R}^3$, let U be the subspace $\{(x, y, z)^T \mid x + y = 0\}$, and W be the subspace $\{(x, y, z)^T \mid x = y = z\}$. Prove that $U \cap W = \{0_V\}$, and $V = U + W$. ***(5 marks)***

(iv) Let V be a vector space, and let $\mathcal{V}$ denote the collection of vectors $v_1, \ldots, v_n$ in V. Define what it means for $\mathcal{V}$ to

(a) be linearly independent;

(b) span V;

(c) be a basis for V. ***(6 marks)***

(v) State the Steinitz Exchange Lemma, and deduce that any two bases for a vector space V have the same number of elements. ***(3 marks)***

(vi) Give a basis for the vector space

$$V = \{A \in M_3(\mathbb{R}) \mid A^T = -A \text{ and } A\mathbf{x} = 0\}$$

where $\mathbf{x} = (1, 1, 1)^T$. ***(3 marks)***

2 (i) Consider the vector space $C(\mathbb{R})$ of continuous functions $\mathbb{R} \longrightarrow \mathbb{R}$. Prove that the functions $\sin x$, $\cos x$ and $\sinh x$ are linearly independent.
(6 marks)

(ii) Consider the vector space $\mathbb{R}[x]_{\leq 3}$ of real polynomials in the variable x of degree at most 3. Give two bases for $\mathbb{R}[x]_{\leq 3}$ with one element in common.
(2 marks)

(iii) Suppose that V is a n-dimensional vector space, with subspaces U and W, and that $V = U \oplus W$. If $\mathcal{U}$ is a basis for U and $\mathcal{W}$ is a basis for W, show that $\mathcal{U} \cup \mathcal{W}$ is a basis for V. (You may assume any standard results on dimensions of subspaces, and that a spanning set for V consisting of n vectors is a basis.) ***(7 marks)***

(iv) Consider the following subspaces of $\mathbb{R}[x]_{\leq 3}$:

$$\begin{aligned} U &= \left\{ f \in \mathbb{R}[x]_{\leq 3} \,\middle|\, \int_{-1}^{1} f(x)\,dx = 0 \right\} \\ W &= \{ f \in \mathbb{R}[x]_{\leq 3} \,|\, f(-1) = f(1) = 0 \} \end{aligned}$$

Find bases for U and W that contain a basis for $U \cap W$. Write down a basis for $U + W$, and deduce that $U + W = \mathbb{R}[x]_{\leq 3}$. ***(10 marks)***

3 (i) Suppose that V and W are vector spaces over $\mathbb{R}$. What does it mean for a map $\phi : V \longrightarrow W$ to be a *linear map*? ***(3 marks)***

(ii) Define the *kernel* of a linear map $\phi : V \longrightarrow W$, and prove that it is a subspace of V. ***(6 marks)***

(iii) Consider the vector space $\mathbb{R}[x]_{\leq 3}$ of real polynomials in the variable x of degree at most 3. Define the linear map $\phi : \mathbb{R}[x]_{\leq 3} \longrightarrow \mathbb{R}[x]_{\leq 3}$ by

$$\phi(f) = x\frac{df}{dx} - f(x+1).$$

(a) Give the matrix of ϕ with respect to the basis $1, x, x^2, x^3$ of $\mathbb{R}[x]_{\leq 3}$.
(4 marks)

(b) What are the eigenvalues of ϕ? ***(2 marks)***

(c) Is the map ϕ surjective? Justify your answer. ***(2 marks)***

(iv) Consider the vector space $V = \left\{ f \in C^{\infty}(\mathbb{R}) \,\middle|\, \frac{d^3 f}{dx^3} + \frac{df}{dx} = 0 \right\}$. You may assume that a basis for V is $1, \cos x, \sin x$. The linear map $\phi : V \longrightarrow \mathbb{R}^2$ is defined by $\phi(f) = (f(0), f(\frac{\pi}{2}))$.

(a) Find a basis for $\ker(\phi)$. ***(2 marks)***

(b) Find bases of V and $\mathbb{R}^2$ with respect to which the matrix of ϕ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. ***(6 marks)***

4 (i) Define the notion of an *inner product* on a finite-dimensional vector space over $\mathbb{R}$. ***(5 marks)***

(ii) Let V be an inner product space (over $\mathbb{R}$). State the Cauchy-Schwarz inequality for vectors $v, w \in V$, including the criterion for when equality holds. ***(3 marks)***

(iii) Consider the inner product space $C[0,1]$ with inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) Show that for any $f \in C[0,1]$, we have

$$\left| \int_0^1 (1+x^2) f(x)\,dx \right| \leq \sqrt{\frac{28}{15}} \sqrt{\int_0^1 f(x)^2\,dx}.$$ ***(6 marks)***

(b) Find an orthogonal basis for the subspace of the inner product space $C[0,1]$ consisting of all polynomials of degree at most 2, using the Gram-Schmidt process on the basis 1, $2x$, $3x^2$. ***(11 marks)***

5 Consider the Fourier inner product space $C[-\pi,\pi]$ of continuous functions $[-\pi,\pi] \longrightarrow \mathbb{R}$ with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt,$$

(i) Compute the cosine of the angle between $\cos 2t$ and $\cos t \cos 3t$. ***(8 marks)***

(ii) Show that if $m < n$ are two positive integers, then $\cos mt \sin t$ and $\cos nt \sin t$ are orthogonal unless $n-m = 2$, and evaluate $\langle \cos mt \sin t, \cos(m+2)t \sin t \rangle$. ***(11 marks)***

(iii) Consider the subspace V of $C[-\pi,\pi]$ of trigonometric polynomials of degree at most 1, spanned by $\{1, \cos t, \sin t\}$. Consider the linear map $D : V \longrightarrow V$ given by $D(f) = \dfrac{df}{dt}$. Compute $\langle D(a + b\cos t + c\sin t), \alpha + \beta\cos t + \gamma\sin t \rangle$.

If $\hat{D}$ denotes the adjoint of D, show that $\hat{D} = -D$. ***(6 marks)***

End of Question Paper