The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2008–9

Continuity and Integration — **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief reasons in all cases, and state the limits if they exist.

$$\frac{4n^2+n+2}{(4n+1)^2}\,, \qquad \left(\frac{1}{6}\right)^{\frac{1}{n}}+\left(\frac{1}{6}\right)^{n}\,, \qquad \frac{n^{35}3^{2n}+n^{3}20^{n}}{2^{4n}+5^{2n}}\,, \qquad (-1)^n\left(\frac{n-1}{n}\right).$$

(12 marks)

(ii) Give the *formal* definition of the notion of a sequence of real numbers *converging to a limit*. Use this definition to prove that the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ converges to a limit. Use the definition of a limit to show that if (x_n) and (y_n) are sequences with $x_n \to 0$ and $y_n \to 0$ then $x_n + y_n \to 0$. ***(13 marks)***

2 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof, provided that they are precisely stated.

(a) Every set with a maximum and a minimum element is finite.

(b) Every non-empty finite set has a maximum and a minimum element.

(c) Every set of rational numbers with a rational lower bound has a rational infimum.

(d) No set has exactly one lower bound.

(e) Every bounded sequence is convergent.

(f) If a convergent sequence has an infinite number of positive terms and an infinite number of negative terms then it converges to zero. ***(25 marks)***

3 (i) For each of the following, give an example of such a function and sketch its graph:

(a) a continuous function $f\colon (0,1) \to \mathbb{R}$ which is not bounded;

(b) a continuous function $g\colon (0,1) \to \mathbb{R}$ which is bounded but has no maximum;

(c) a function $h\colon [0,1] \to \mathbb{R}$ which is not bounded;

(d) a function $k\colon [0,1] \to \mathbb{R}$ which is bounded but has no maximum.

State a theorem which guarantees that certain functions defined on an interval have a maximum and a minimum. Why is it that none of the above examples you have given contradicts this theorem? ***(15 marks)***

(ii) If f is a real-valued function, give the definition of the *derivative* of f. Use the definition to find the derivative of $f(x) := x^n$ where n is a positive integer.
Prove that the function $k(x) := |x|$ is not differentiable at $x = 0$. ***(10 marks)***

4 (i) State the *Intermediate Value Theorem* (IVT) and use it to show that the equation $2x^2(x+4)-1 = 0$ has a root in each of the intervals $(-4,-3)$, $(-1,0)$ and $(0,1)$, stating carefully how the IVT is used. Consider the root in $(0,1)$; use two iterations of the bisection method to find a smaller interval in which this root lies. ***(10 marks)***

(ii) Let $f : [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on (a,b). Let $g : [a,b] \to \mathbb{R}$ be defined by the equation

$$g(x) = f(x) - \alpha x,$$

where α is the *unique* real number determined by the condition $g(a) = g(b)$. Express α in terms of a, b, $f(a)$ and $f(b)$. State the Mean Value Theorem and show how it can be proved by applying Rolle's Theorem to the function g. ***(10 marks)***

Apply the Mean Value Theorem to the function $f(x) = \tan^{-1} x$ on the interval $[0,b]$, where $0 < b$, to show that

$$\frac{b}{1+b^2} < \tan^{-1} b < b.$$

[**Hint**: You may use the fact that $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$.] ***(5 marks)***

5 (i) Consider the function $f(x) := b^2 - x^2$, and the partition $P_N := \{0 = x_0 < \ldots < x_N = b\}$ of the interval $[0, b]$ into N equally sized pieces, so $x_i = \frac{ib}{N}$. Define what is meant by the *lower sum* $L_{P_N}(f)$. Draw a picture to illustrate the lower sum, and find an explicit formula for it. Calculate the limit $\lim_{N\to\infty} L_{P_N}(f)$.

[**Hint**: You may use the formula $\sum_{i=1}^{N} i^2 = \frac{1}{6}N(N+1)(2N+1)$.] ***(14 marks)***

(ii) Prove that if $f(x)$ is a *decreasing* function on the interval $[a, b]$ and P is a partition of $[a, b]$ into N equal pieces then

$$U_{P_N} - L_{P_N} = \frac{1}{N}(b-a)(f(a) - f(b)).$$

Use this to deduce that such a decreasing function is Riemann integrable on the interval $[a, b]$. ***(8 marks)***

(iii) Hence, or otherwise, prove that the function in part (i) is Riemann integrable on $[0, b]$ and state the value of the integral. ***(3 marks)***

End of Question Paper