The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2008–2009

FOUNDATION YEAR MATHEMATICS II — 3 hours

You should attempt ***all*** *questions in Section A.*
You should attempt ***three*** *questions in Section B. You are advised* ***not*** *to answer more than three questions: if you do, only your best three will be counted. Section A carries 55 marks and each question in Section B is worth 15 marks.*

Section A

A1 A surveyor on horizontal ground observes a building 30° west of north. After walking 400 m north he observes that the same building is at 60° west of north. How much further must he walk *due north* such that the building is to his west? At this location, how far away is the building? Give your answers correct to the nearest metre. ***(7 marks)***

A2 Find

(i) the equation of the line L_1 passing through the points $(-3, 0)$ and $(2, -10)$,

(ii) the equation of the line L_2 which is perpendicular to L_1 and which passes through the point $(2, 0)$,

(iii) the point of intersection of L_1 and L_2. ***(8 marks)***

A3 Given that $\mathbf{a} = (1, 2, 0)$, $\mathbf{b} = (0, 3, 3)$ and $\mathbf{c} = (1, 6, 1)$ calculate

(i) the scalar product $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c}$,

(ii) the vector product $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$,

(iii) the unit vector in the direction of $\mathbf{a} + \mathbf{c}$. ***(9 marks)***

A4 In the triangle OAB, the point D is the midpoint of AB. Given that $\mathbf{a} = \vec{OA}$ and $\mathbf{b} = \vec{OB}$:

(i) Express $\vec{AD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and hence show that

$$\vec{OD} = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

(ii) Evaluate $\vec{AD} \cdot \vec{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and hence show that

$$\vec{AD} \cdot \vec{OD} = \frac{1}{4}(b^2 - a^2),$$

where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. ***(7 marks)***

A5 (i) Find *all* the solutions to the equation

$$\tan 2\theta = \sqrt{3},$$

where $-180^\circ \leq \theta \leq 180^\circ$.

(ii) Use the result $\sin(A + B) = \sin A \cos B + \cos A \sin B$ to show that

$$\sin 105^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

(9 marks)

A6 (i) An arithmetic series has 29 as its 7th term and 73 as its 18th term. Find the sum of its first 40 terms.

(ii) A geometric series is given by

$$1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \cdots.$$

Determine the sum of the first n terms.
Decide whether or not this series converges to a finite limit, and, if it does, find this limiting value. ***(9 marks)***

A7 (i) Complex numbers z_1 and z_2 are given by

$$z_1 = 7 + i, \qquad z_2 = 3 - 5i.$$

Evaluate $\frac{z_1}{z_2}$ giving your answer in the form $a + ib$, where a and b are real numbers.

(ii) Find the modulus and argument of the complex number

$$z = \sqrt{5} + 2i.$$

(6 marks)

Section B

B1 Find the vector equation of the line L_1 passing through the point $(-2, 7, 2)$ in the direction of the vector $(-5, 6, -5)$. ***(1 mark)***

Show that the equation of the line L_2 passing through both $(5, -2, 3)$ and $(8, -5, 12)$ may be written as

$$\mathbf{r}(t) = (5, -2, 3) + t(3, -3, 9).$$

(3 marks)

Show that the lines L_1 and L_2 intersect and find the coordinates of the point of intersection. ***(11 marks)***

B2 (i) Determine the angle between the two vectors

$$\mathbf{a} = (1, \sqrt{2}, 0) \text{ and } \mathbf{b} = (0, 1, -1).$$

Find a unit vector perpendicular to the plane containing **a** and **b**.

(10 marks)

(ii) Given that **i**, **j** and **k** are the unit vectors in the directions of the coordinate axes Ox, Oy and Oz respectively, evaluate

(a) $\mathbf{i} \cdot \mathbf{j}$,

(b) $\mathbf{i} \times \mathbf{j}$,

(c) $(\mathbf{i} \times \mathbf{j}) \cdot \mathbf{k}$. ***(5 marks)***

B3 (i) Find the centre and radius of the circle

$$x^2 + y^2 - 4x - 6y - 12 = 0.$$

(4 marks)

(ii) Calculate the distance between the centre of the circle and the point $(0, 0)$. ***(3 marks)***

(iii) Find the equation of the tangent to the circle at the point $(5, -1)$. ***(8 marks)***

B4 (i) Use the Newton-Raphson method to find the positive real root of the equation

$$2x - 2 + \sin x = 0,$$

where x is measured in radians, correct to 3 decimal places. Take the initial guess to be $x_0 = 1$. ***(9 marks)***

(ii) Simplify

$$\frac{15!}{11!4!} + \frac{15!}{12!3!},$$

leaving your answer in terms of factorials. ***(6 marks)***

B5 (i) Find the general solution of

$$\frac{d^2y}{dx^2} = 0.$$

Find the particular solution if

$$\frac{dy}{dx} = 2 \text{ and } y = -3 \text{ when } x = 1.$$

(6 marks)

(ii) Form the differential equation for which the expression

$$r = A\sin\theta$$

is a valid solution, where A is a constant. ***(4 marks)***

(iii) Verify that $y = Ae^{2x} + Be^{-x}$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

(5 marks)

End of Question Paper