The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2008 - 2009 **2 hours**

APPLIED DIFFERENTIAL EQUATIONS

*Marks will be awarded for your best **FOUR** answers.*

1.(i) The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx}=1-x-4y$$

and the condition $y=1$ at $x=0$. Use the third-order Runge-Kutta method (Heun's method) given below with step-length $h=0.2$ to determine y at $x=0.2$. Work throughout with four decimal places.

(9 marks)

Heun's method for solving $dy/dx=f(x,y)$ *is given by*

$$k_1=hf(x_n,y_n)$$
$$k_2=hf(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1)$$
$$k_3=hf(x_n+h,y_n-k_1+2k_2)$$
$$y_{n+1}=y_n+\frac{1}{6}(k_1+4k_2+k_3)$$

Question 1 continued on page 2

Question 1 continued from page 1

(ii) The following results were obtained by using Heun's method to solve the equation of part (i) as far as $x=2$ with two different step-lengths:

h	$y(2)$
0.04	- 0.187270
0.20	- 0.187334

Use this data to estimate the step-length required to ensure that the global error in $y(2)$ does not exceed 5×10^{-8} .

(8 marks)

(iii) Apply Heun's method to the test equation $dy/dx=\lambda y$ and hence show that

$$y_{n+1}=R(\bar{h})\,y_n \quad ,$$

where $R(\bar{h})$ is a cubic polynomial in $\bar{h}$ and $\bar{h}=h\lambda$.

(8 marks)

2. (i) The function $y(x)$ satisfies the differential equation

$$y'(x) = y^3 + x^2 - 4 \ ,$$

and initial condition $y(1) = 1$. Given the following additional data $y(1.05) = 0.895$, $y(1.10) = 0.782$, correct to 3 decimal places, advance the solution to $x = 1.15$ by applying the predictor-corrector method

$$y_{n+1}^{P} = y_n + \frac{h}{12}[23 f_n - 16 f_{n-1} + 5 f_{n-2}] \qquad (1)$$

$$y_{n+1}^{C} = y_n + \frac{h}{12}[5 f_{n+1} + 8 f_n - f_{n-1}] \qquad (2)$$

with $h = 0.05$ where y_i is an estimate of $y(x_i)$ and $f_i = f(x_i, y_i) = y_i^3 + x_i^2 - 4$. Work throughout correct to 3 decimal places.

(8 marks)

(ii) Given that the truncation errors associated with formulae (1) and (2) are, respectively

$$\frac{3}{8} h^4 f^{(3)}(\xi_1), \quad x_{n-2} \le \xi_1 \le x_n, \quad \text{and} \quad -\frac{1}{24} h^4 f^{(3)}(\xi_2), \quad x_{n-1} \le \xi_2 \le x_{n+1},$$

show that

$$y(x_{n+1}) - y_{n+1}^{C} \approx \frac{1}{10}\left[y_{n+1}^{P} - y_{n+1}^{C} \right] \ ,$$

where $y(x_{n+1})$ denotes the true solution at $x = x_{n+1}$. State clearly any assumptions that you make.

(7 marks)

(iii) Determine the values of α_j, β_j, $j = 0, 1, 2$ for formula (2) when this formula is written in the form

$$\sum_{j=0}^{2} \alpha_j y_{n-1+j} = h \sum_{j=0}^{2} \beta_j f_{n-1+j} \ .$$

(5 marks)

Given that the general form of the truncation error may be written as

$$T = C_0 y(x) + C_1 h y'(x) + \ldots$$

where $C_0 = \alpha_0 + \alpha_1 + \alpha_2$ and $C_1 = \alpha_1 + 2\alpha_2 - (\beta_0 + \beta_1 + \beta_2)$, evaluate C_0 and C_1 and state whether formula (2) is consistent or inconsistent with the differential equation giving reasons for your answer.

(5 marks)

3. (i) Use Taylor series to derive the approximation

$$y_n' = \frac{y_{n+1} - y_{n-1}}{2h}$$

where y_{n+1} is an estimate of $y(x_n + h)$, and show that the leading term of the error associated with this approximation is $O(h^2)$.

(5 marks)

Use the above relation and the approximation $y_n'' = (y_{n+1} - 2y_n + y_{n-1})/h^2$ to show that the differential equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = R(x)$$

may be approximated at $x = x_n$ by the relation

$$(1 - \frac{1}{2}hP_n)y_{n-1} - (2 - h^2 Q_n)y_n + (1 + \frac{1}{2}hP_n)y_{n+1} = h^2 R_n,$$

where $P_n = P(x_n)$, $Q_n = Q(x_n)$ and $R_n = R(x_n)$.

(3 marks)

Use the above relation with $h = 0.2$ to show that the differential equation

$$y'' + e^x y' + x^2 y = x$$

with boundary conditions $y(0) = 0$, $y(1) = 1$ may be approximated by a system of linear algebraic equations $A\boldsymbol{y} = \boldsymbol{b}$ where $\boldsymbol{b} = (0.008, 0.016, 0.024, -1.191)^T$. Determine the elements of A correct to 4 decimal places.

(10 marks)

(ii) The following table contains grid-point values of two solutions $Y_1(x)$ and $Y_2(x)$ of a linear differential equation $d^2y/dx^2 = f(x, y, y')$ obtained using the fourth-order Runge-Kutta method. $Y_1(x)$ was determined using the initial conditions $y(1) = 1$, $y'(1) = 0$ and $Y_2(x)$ was obtained using $y(1) = 1$, $y'(1) = 1$.

x	1.25	1.5	1.75	2.0
$Y_1(x)$	1.07444	1.36280	2.02865	3.41281
$Y_2(x)$	1.36222	2.05472	3.35550	5.85592

Form a linear combination of these two solutions which is the numerical solution to the equation $d^2y/dx^2 = f(x, y, y')$ with boundary conditions $y(1) = 1$, $y(2) = 2$. Calculate values of this solution at all the *x*-values in the Table.

(7 marks)

4.(i) By writing $y_1 = y$ and $y_2 = dy_1/dx$ show that the second-order equation

$$\frac{d^2 y}{d x^2} = f(x, y, dy/dx)$$

may be written in the form

$$\frac{d\boldsymbol{Y}}{dx} = \boldsymbol{F}(x, \boldsymbol{Y})$$

where $\boldsymbol{Y} = [y_1, y_2]^T$ and $\boldsymbol{F}(x, \boldsymbol{Y}) = [y_2\ , f(x, y_1, y_2)]^T$.

(4 marks)

The function $y(x)$ satisfies the equation

$$y'' + x y' + x y = 1$$

and the conditions $y(0) = y'(0) = 0$. Use Heun's method (given for a single equation in Question 1) to determine y and y' at $x = 0.1$ using a step-length $h = 0.1$.

(9 marks)

(ii) By using the change of independent variables $\nu = x + at$, $\eta = x + bt$, where *a* and *b* are constants, show that the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (3)$$

may be written as

$$\frac{\partial^2 u}{\partial \nu \partial \eta} = 0 \quad .$$

Hence determine the general solution of equation (3).

(12 marks)

5.(i) The function $u(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (4)$$

in the rectangular region $0 \le x \le a$, $0 \le y \le b$ together with the boundary conditions

$$\frac{\partial u(0, y)}{\partial x}=0, \quad \frac{\partial u(a, y)}{\partial x}=0, \quad (0 \le y \le b)$$

$$u(x, 0)=0 \quad (0 \le x \le a).$$

Show that solutions of (4) of the form $X(x)Y(y)$ satisfy the relation

$$\frac{X''}{X}=-\frac{Y''}{Y}=\alpha \quad ,$$

where α is a constant.

(4 marks)

(ii) Assuming that $\alpha=-s^2$ $(s \neq 0)$, show that there is a solution of equation (4) which satisfies the above boundary conditions of the form

$$X(x)Y(y)=A\cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a},$$

where A is an arbitrary constant.

(10 marks)

(iii) Show that the corresponding solution for $\alpha=0$ is

$$X(x)Y(y)=C y,$$

where C is an arbitrary constant.

(5 marks)

(iv) Given that $u(x, y)$ also satisfies the boundary condition

$$u(x, b)=u_0\cos\frac{2\pi x}{a} \qquad (0 \le x \le a)$$

where u_0 is a constant, determine the solution of equation (4) which satisfies all four boundary conditions.

(6 marks)

Note: You may assume that only trivial solutions result from the choice $\alpha > 0$.

END OF QUESTION PAPER