Data Provided: Formula Sheet

AMA242

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS   Autumn Semester 2008-2009

Mathematics III(Electrical)   2 hours

*Attempt **ALL** questions.*

1 For the function $w = 1/z$ find the image of the following in the w-plane. Sketch you results in the z and w-planes. For (ii) and (iii) $0 \le r < \infty$, and for (iv) $0 \le \theta < 2\pi$.

(i) The point $z = 1 - j$. ***(3 marks)***

(ii) The line $w = re^{-jz/6}$. ***(5 marks)***

(iii) The line $z = -1 + re^{j\pi/2}$. ***(13 marks)***

(iv) The circle $z = 2e^{j\theta}$. ***(4 marks)***

2 (i) Find the expansion in the Taylor series of the function $\dfrac{1}{1+z}$ about $z = 2j$. Show the region of convergence on the Argand diagram and indicate the pole that determines the radius of convergence. ***(10 marks)***

(ii) Find the Laurent series expansion of $\dfrac{1}{2j + (2-j)z - z^2}$ in the region $1 < |z| < 2$. ***(15 marks)***

3 (i) Find all the poles of $f(z) = \dfrac{z^4}{z^4 + 13z^2 + 36}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)dz$, writing your solutions in the form $a + jb$, where a and b are real, where

(a) C is the circle $|z| = 4$

(b) C is the circle $|z + 2j| = 2$.

(13 marks)

(ii) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2}\, dx$$

(Hint: take $\cos x = \mathrm{Re}(e^{jx})$, where Re indicates the real part, and first calculate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{jx}}{1 + x^2}\, dx$) ***(12 marks)***

4 (i) The functions $x(t)$ and $y(t)$ satisfy the system of differential equations

$$\begin{cases} \dot{x} = x + 2y, \\ \dot{y} = y + 2x \end{cases}$$

(where dot denotes differentiation with respect to t) and the initial conditions $x(0) = 4$ and $y(0) = 2$. Use the Laplace transform to find $x(t)$ and $y(t)$ for $t > 0$. ***(15 marks)***

(ii) Two functions, $f(t)$ and $g(t)$, are defined by

$$f(t) = \begin{cases} e^{t}, & t \le 0, \\ e^{-2t}, & t > 0, \end{cases} \qquad g(t) = \begin{cases} e^{2(t-1)}, & t \le 1, \\ e^{-3(t-1)}, & t > 1. \end{cases}$$

(a) Using direct integration, calculate the Fourier transforms, $F(\omega)$ and $G(\omega)$, of $f(t)$ and $g(t)$.

(b) The convolution of the functions $f(t)$ and $g(t)$ is the function $h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau$. Use the convolution theorem to calculate the Fourier transform $H(\omega)$ of the function $h(t)$. ***(10 marks)***

End of Question Paper

FORMULA SHEET

Table of Laplace Transforms

$f(t)$	$F(s)$	Region of validity
constant $= c$	$\frac{c}{s}$	$Re(s) > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$Re(s) > \alpha$
t	$\frac{1}{s^2}$	$Re(s) > 0$
$\cos kt$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
t^n	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$	$Re(s) > \alpha$
$e^{\alpha t} \sin kt$	$\frac{k}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$e^{\alpha t} \cos kt$	$\frac{s-\alpha}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$\delta(t-T)$	e^{-sT}	delta function
$H(t-T)$	$\frac{e^{-sT}}{s}$	step function
$H(t) - H(t-T)$	$\frac{1}{s}(1 - e^{-sT})$	rectangular pulse

Note: in this table the parameters α and k are real constants and H is the Heaviside step function.

Some general properties of the Laplace transform

In the following table the notation $\mathbf{L}\{f(t)\} = F(s)$ has been used.

$\mathbf{L}\{af(t) + bg(t)\} = a\mathbf{L}\{f(t)\} + b\mathbf{L}\{g(t)\}$	linearity
$\mathbf{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathbf{L}\{\frac{d^2}{dt^2}f(t)\} = s^2F(s) - sf(0) - f'(0)$	differentiation twice with respect to t
If $g(t) = \int_0^t f(u)du$ then $\mathbf{L}\{g(t)\} = \frac{1}{s}F(s)$	integration
$\mathbf{L}\{tf(t)\} = -\frac{dF}{ds}$	differentiation w.r.t. s
$\mathbf{L}\{e^{-kt}f(t)\} = F(k + s)$	shift
$\mathbf{L}\{f(at)\} = \frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	scaling
$\mathbf{L}\{f(t - a)H(t - a)\} = e^{-as}F(s)$	time delay

Convolution

For causal functions

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau$$

and has Laplace transform $F(s)G(s)$.

Fourier transform

The Fourier transform $\mathbf{F}(\omega)$ of a function $f(t)$ is defined by

$$\mathbf{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt.$$

The time shift property: the Fourier transform of a function $f(t - T) = e^{-j\omega T}\ \mathbf{F}(\omega)$.

The scaling property: the Fourier transform of a function $f(at) = \frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$.

Residues

The general formula for the residue at a pole, z_0, of order m is

$$\frac{1}{(m-1)!} lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]\right).$$