The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2008–2009**

Nonlinear Mathematics **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper I denotes an open interval in $\mathbb{R}$ and U denotes an open subset of $\mathbb{R}^2$.

1 (i) Let $f\colon\ I \to \mathbb{R}$ be a function and let $a \in I$.

Define the meaning of the expressions

$$\lim_{x\uparrow a} f(x) = L \qquad \text{and} \qquad \lim_{x\downarrow a} f(x) = L.$$

(6 marks)

(ii) Define $f\colon\ (-1,1) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1-\sqrt{1-x^2} & -1<x<0 \\ 0 & x=0 \\ -1+\sqrt{1-x^2} & 0<x<1 \end{cases}$$

(a) Show that f is differentiable at $x=0$, and find $f'(0)$, stating without proof any theorem from the course which you use. **(8 marks)**

(b) Use calculus to write down $f'(x)$ for $x \in (-1,0)$ and $x \in (0,1)$. **(4 marks)**

(c) Determine whether or not $f'(x)$ is differentiable at $x=0$. **(7 marks)**

2 (i) (a) Define what it means for a sequence (x_n, y_n) in $\mathbb{R}^2$ to converge to a point (a, b). **(2 marks)**

(b) Let (x_n, y_n) be a sequence in $\mathbb{R}^2$ and let (a, b) be a point.

Prove that if

$$|(x_n, y_n) - (a, b)| \to 0$$

as $n \to \infty$, then $(x_n, y_n) \to (a, b)$. **(8 marks)**

(ii) Let $F\colon U \to \mathbb{R}$ be a differentiable function defined on an open set $U \subseteq \mathbb{R}^2$ and let $g\colon I \to \mathbb{R}$ be a differentiable function defined on an open interval $I \subseteq \mathbb{R}$, such that $F(x, y) \in I$ for all $(x, y) \in U$.

State the Chain Rule for $g \circ F$: (a) as an equation for $D(g \circ F)(x, y)$, and (b) as an equation for the partial derivatives. **(6 marks)**

(iii) Let $\Omega\colon \mathbb{R} \to \mathbb{R}$ be the function

$$\Omega(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leqslant 0. \end{cases}$$

It was proved in lectures that Ω is differentiable and that $\Omega'(0) = 0$. Using calculus to find $\Omega'(t)$ for $t \neq 0$, apply (ii) to find $D(H)(x, y)$ where

$$H(x, y) = \begin{cases} e^{-\frac{1}{1-x^2-y^2}} & \text{when } x^2 + y^2 < 1 \\ 0 & \text{when } x^2 + y^2 \geqslant 1. \end{cases}$$

(9 marks)

3 (a) Let $(a,b) \in \mathbb{R}^2$ and let $r > 0$. Define the open ball $B((a,b),r)$.

Define what it means for a set $U \subseteq \mathbb{R}^2$ to be open. **(4 marks)**

(b) Let $U_1 \subseteq \mathbb{R}^2$ and $U_2 \subseteq \mathbb{R}^2$ be any open sets. Prove that the intersection $U_1 \cap U_2$ is open. **(7 marks)**

(c) For a function $F\colon U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^2$ is a set, and $I \subseteq \mathbb{R}$ is an interval, define the set $F^{-1}(I)$.

State carefully (but do not prove) the Theorem from lectures which ensures that $F^{-1}(I)$ is an open set. **(4 marks)**

(d) Using (b) and (c) if you wish, prove that the set V shown in Figure 1 is open.

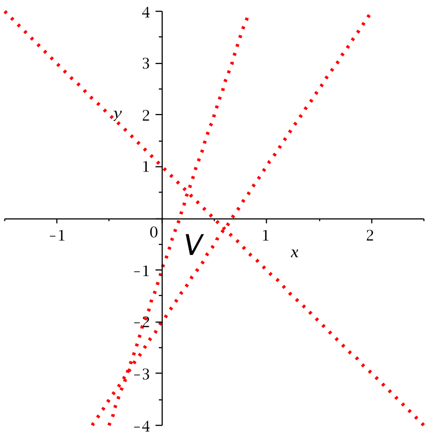

Figure 1: For Question 3(d). The set V is the region bounded by the three dotted lines. The lines have equations $y = 3x - 2$, $y = 1 - 2x$ and $y = 6x - 1$.

(10 marks)

4 (i) Define what it means for a function $F\colon U \to \mathbb{R}$ defined on an open set $U \subseteq \mathbb{R}^2$ to be differentiable at a point $(a,b) \in U$. **(4 marks)**

(ii) Prove that the function F defined by

$$F(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0), \\ 0 & \text{for } (x,y) = 0, \end{cases}$$

is differentiable at $(0,0)$. **(5 marks)**

(iii) Let $L\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear function. Show that there is a constant C such that

$$|L(\lambda,\mu)| \leqslant C\,|(\lambda,\mu)|$$

for all $(\lambda,\mu) \in \mathbb{R}^2$. **(4 marks)**

(iv) Let $F\colon U \to \mathbb{R}$ be defined on an open set $U \subseteq \mathbb{R}^2$ and assume that it is differentiable at (a,b).

(a) Show that there exists $r > 0$ such that if $|(\lambda,\mu)| < r$ then

$$|F(a+\lambda,\, b+\mu) - F(a,b) - D(F)(a,b)(\lambda,\mu)| \leqslant |(\lambda,\mu)|.$$

(6 marks)

(b) Using (iii) and (iv)(a) or otherwise, prove that there exists $r > 0$ and $K > 0$ such that

$$\text{if} \quad |(\lambda,\mu)| < r \quad \text{then} \quad |F(a+\lambda,\, b+\mu) - F(a,b)| \leqslant K|(\lambda,\mu)|.$$

(6 marks)

5 (a) Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map $\varphi(x,y,z) = (yz,\, zx,\, xy)$.

Determine the rank of φ at each point of $\mathbb{R}^3$. **(10 marks)**

(b) Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ be the unit sphere, centre origin, in $\mathbb{R}^3$. Show that the image $\varphi(S)$ of S under φ is a subset of

$$M = \{(u,v,w) \in \mathbb{R}^3 \mid u^2v^2 + v^2w^2 + w^2u^2 - uvw = 0\}.$$

(4 marks)

(c) Let $G\colon \mathbb{R}^3 \to \mathbb{R}$ be the map $G(u,v,w) = u^2v^2 + v^2w^2 + w^2u^2 - uvw$.

Show that the rank of G is zero on each of the coordinate axes, and that the rank is 1 at all other points of M. **(11 marks)**

End of Question Paper