The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2009–2010

FOUNDATION YEAR MATHEMATICS II — 3 hours

*You should attempt **all** questions.*

1 (i) A surveyor on horizontal ground observes a tower in the distance. He notes that the angle of elevation of the top of the tower is 10° from the horizontal. After walking 150 m towards the tower he notes that it has an angle of elevation of 15°. Draw a diagram of the situation and find the height of the tower, giving your answer correct to the nearest metre.

(7 marks)

(ii) Give the formula for the gradient of a line passing through the points (x_1, y_1) and (x_2, y_2) relative to the axes Oxy. Given that the line L passes through the points $A = (8, 1)$ and $B = (2, 3)$, find

(a) the gradient of L,

(b) the equation of L, and

(c) the gradient of a line perpendicular to L. ***(8 marks)***

(iii) Find the vector equation of the line L_1 passing through the point $(3, -1, -3)$ in the direction of the vector $(-2, 4, 12)$.
Find the vector equation of the line L_2 passing through both $(1, 2, 1)$ and $(6, -3, 11)$.
Find the coordinates of the point of intersection of the lines L_1 and L_2. ***(10 marks)***

2 (i) State the geometric interpretations of

(a) the scalar product and

(b) the vector product

taking care to explain any notation that you use. Explain briefly why the expression $\mathbf{a} \times (\mathbf{c} \cdot \mathbf{d})$ does not make sense and why $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

(6 marks)

(ii) The points A, B and C lie on a circle centred at the origin O and AOC is a diameter of the circle. Given that $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B relative to O, draw a diagram to illustrate the situation and express $\overrightarrow{BA}$, $\overrightarrow{BC}$ and $\overrightarrow{BA} \cdot \overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence show that ABC is a right angle. ***(8 marks)***

(iii) Given that $\mathbf{a} = (1, -2, 0)$ and $\mathbf{b} = (1, 2, 2)$, calculate

(a) the scalar product $\mathbf{a} \cdot \mathbf{b}$,

(b) the angle between $\mathbf{a}$ and $\mathbf{b}$,

(c) the vector $\mathbf{a} \times (\mathbf{a} + \mathbf{b})$. ***(8 marks)***

(iv) Given that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the directions of the coordinate axes Ox, Oy and Oz respectively, evaluate

(a) $\mathbf{i} + (\mathbf{j} \times \mathbf{k})$,

(b) $\mathbf{i} \times \mathbf{j}$. ***(3 marks)***

3 (i) You are given that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

(a) Derive an expression for $\sin 2A$ in terms of $\sin A$ and $\cos A$.

(b) Hence derive an expression for $\sin 2A$ in terms of $\tan A$. ***(6 marks)***

(ii) Find the terms up to and including x^3 in the infinite binomial expansion of $(1+3x)^{\frac{1}{3}}$, stating the range of values for which your expansion is valid. ***(5 marks)***

(iii) Find the sum of the first 10 terms of the geometric progression

$$3 + 6 + 12 + 24 + \ldots,$$

using the appropriate formula. ***(4 marks)***

(iv) Complex numbers z_1 and z_2 are given by $z_1 = 5 + i$ and $z_2 = 3 - 5i$.

(a) Evaluate $z_1 z_2$ giving your answer in the form $a + ib$, where a and b are real numbers.

(b) Evaluate $\frac{z_1}{z_2}$ giving your answer in the form $a + ib$, where a and b are real numbers.

(c) Find the modulus and argument of the complex number $z = 2 + i$. ***(10 marks)***

4 (i) Use the Newton-Raphson method to find the positive real root of the equation $x^2 - 5 = 0$ correct to 2 decimal places. Take the initial iterate to be $x_0 = 2$. ***(5 marks)***

(ii) Simplify $\frac{n!}{(n-2)!}$. ***(4 marks)***

(iii) Express the recurring decimal $0.311\dot{1}$ as a fraction. ***(5 marks)***

(iv) Caesium 137 decays at a rate equal to k times the amount of Caesium present at any time, where $k = 0.625 \times 10^{-9}\ \mathrm{s}^{-1}$.

(a) Formulate the differential equation that models the decay process. ***(4 marks)***

(b) As a result of the accident in Chernobyl in 1986, Caesium 137 was deposited on upland parts of the United Kingdom. Find the general solution of the equation resulting from part (i) and use your solution to estimate the year by which half of the Caesium 137 will have decayed.

[You may assume that, approximately, 1 year = 3×10^7 s.] ***(7 marks)***

End of Question Paper