



Attempt all the questions. The allocation of marks is shown in brackets.

- 1 (i) Solve the differential equation

$$y'' + 3y' + 2y = 2x^2 + 1 - e^{-x}, \quad y = y(x)$$

subject to the conditions  $y(0) = 1$  and  $y'(0) = 1$ . (17 marks)

- (ii) Solve the first order differential equation using the integrating factor technique

$$\frac{dy}{dt} = 2y + 4 - t$$

subject to the condition  $y(0) = -2$ . Discuss the behaviour of the solution as  $t \rightarrow \infty$ . (8 marks)

- 2 (i) The process of a radioactive decay is governed by the differential equation

$$\frac{dQ}{dt} = -rQ$$

where  $r > 0$  is constant. Find the solution of the above equation and use the result to find a solution for the following problem: a radioactive isotope has a half-life of 16 days. You wish to have 30 g at the end of 30 days. How much radioisotope should you start with? Work correct to 3 decimal places. (8 marks)

- (ii) The evolution of a population exposed to a constant harvesting is given by

$$\frac{dx}{dt} = x(7 - x) - 6, \quad y = y(t)$$

Define the notion of *equilibrium level*. Find the equilibrium points (levels) of the above equation and discuss their nature. (8 marks)

2 (continued)

(iii) The equation of a damped oscillator is given by

$$\frac{d^2y}{dt^2} + \beta \frac{dy}{dt} + 4y = 0$$

where the quantity  $\beta$  is a real and positive constant and it describes the damping mechanism. Find the interval of  $\beta$  so that the solution of the differential equation describes oscillations with decaying amplitude.

(9 marks)

3 (i) Solve the separable differential equation

$$\frac{dy}{dx} = \frac{y-1}{x^2-3x-4}, \quad y = y(x),$$

subject to the condition  $y(0) = 0$ .

(15 marks)

(ii) Solve the Euler-Cauchy-type differential equation

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0$$

subject to the initial conditions  $y(1) = 4$  and  $y'(1) = -2$ .

(10 marks)

4 (i) Solve the second order difference equation

$$u_{n+2} - 3u_{n+1} + \frac{9}{4}u_n = n^2 + 1$$

subject to the conditions  $u_0 = 100$  and  $u_1 = 1$ .

(10 marks)

(ii) Find the integrating factor for the differential equation

$$e^{-x} \frac{dy}{dx} + 3ye^{-x} - 1 = 0, \quad y = y(x)$$

and hence obtain the solution of the differential equation which satisfies  $y(0) = 1$ .

(10 marks)

(iii) Find the difference equation satisfied by

$$(a) u_n = A \times 2^n + B \times (-5)^n, \quad (b) u_n = (A + Bn)(-7)^n$$

(5 marks)

End of Question Paper