Data provided: Formula sheet

The University Of Sheffield.

MAS165

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2009-2010

Mathematics for Physicists — 2 hours

You should attempt ALL questions of this exam.

Section A

A1 A plane is given by the equation

$$4x + 5y + 7z = 21$$

and a line by the equation $\mathbf{r} = (1, 2, 3) + \lambda(1, 2, -2)$, where λ is a real parameter.

(i) Show that the line does not intersect the plane. ***(4 marks)***

(ii) Therefore, calculate the distance of the line to the plane. ***(4 marks)***

(iii) Find the direction of the line of intersection of the two planes $x+3y-z = 5$ and $2(x - y) + 4z = 3$. ***(5 marks)***

A2 Let $f(t) = t^3 - t^2$ be a scalar function and $\mathbf{V} = (1/t, t^2, t^3)$ and $\mathbf{W} = (t, sin(t), 0)$ be vectors. Find $d(f\mathbf{V})/dt$, $d(\mathbf{V} \cdot \mathbf{W})/dt$ and $d(\mathbf{V} \times \mathbf{W})/dt$. ***(8 marks)***

A3 Stokes' theorem may be written:

$$\oint_C \mathbf{G}.\mathbf{dr} = \int_S (\nabla \times \mathbf{G}).\hat{\mathbf{n}}\, dS$$

Indicate whether the following statements about Stokes' theorem, as expressed here, are true or false

(i) The term $(\nabla \times \mathbf{G})$ is the curl of the vector field $\mathbf{G}$.

(ii) The surface S is surrounded by a closed line C.

(iii) $\hat{\mathbf{n}}$ is a unit vector parallel with the boundary C.

(iv) $\int_S dS$ is a surface integral, over the surface S.

(4 marks)

Section B

B1 (i) Consider the function

$$f(x,y) = \tan^{-1}\frac{y}{x}$$

Determine all partial derivatives to first and second order and show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Hint: $\tan^{-1}$ (also called arctan) is the inverse function of the tan-function and you are given that

$$\frac{d\tan^{-1} u}{du} = \frac{1}{1+u^2}.$$

(12 marks)

(ii) A scalar function is given as

$$\phi(x,y,z) = x^2 - y\sin(x-z).$$

(a) Calculate the gradient of $\phi(x,y,z)$, i.e. calculate $\mathbf{V} = \nabla\phi$. ***(3 marks)***

(b) Using your result, calculate the divergence of $\mathbf{V}$. ***(4 marks)***

(c) By explicit calculation, show that $\nabla \times \mathbf{V} = 0$. ***(6 marks)***

B2 (i) A vector field is given by

$$\mathbf{V} = r^2\hat{\mathbf{r}} + (a + r)\hat{\boldsymbol{\theta}} + bz\hat{\mathbf{z}}$$

in cylindrical polar coordinates, where a and b are positive constants. Calculate the divergence and curl of the vector field, given that the divergence and curl may be expressed in cylindrical coordinates as

$$\nabla.\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_2) + \frac{\partial}{\partial z}(V_3)$$

and

$$\nabla \times \mathbf{V} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_1 & rV_2 & V_3 \end{vmatrix}$$

respectively. Indicate where the field is irrotational. ***(10 marks)***

(ii) Sketch the region of integration represented by the repeated integral

$$\int\int_R xy^2 \, dx \, dy$$

where R is the region such that $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq a^2$. By transforming to plane polar coordinates, evaluate the integral.

(15 marks)

B3 (i) A particle P with position vector $\mathbf{r}$ moves in a plane polar (r, θ) coordinate system. Write down the relationship between the unit vectors $\mathbf{i}, \mathbf{j}$ in Cartesian coordinates (x, y) and the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ in plane polar coordinates (r, θ). Hence, or otherwise, show that the velocity $\mathbf{v}$ of the particle may be expressed as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

and find the component of the acceleration of the particle in the $\hat{\boldsymbol{\theta}}$ direction.

(15 marks)

(ii) A magnetic field is given, in cylindrical polar coordinates (r, θ, z), as $\mathbf{H} = H_0 r^2 \hat{\theta}/a^2$, with $r \leq a$, where H_0 and a are positive constants. The magnetic field vanishes for $r > a$. Evaluate

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r},$$

where $\mathcal{C}$ is the circle $z = 0$, $r = R$, described in the anticlockwise sense for $R < a$.

(10 marks)

End of Question Paper