The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2009–10

Advanced Calculus — **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) The random variables X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 6x^2y & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that

$$\int_0^1 \int_0^1 f_{X,Y}\, dx\, dy = 1 .$$

(b) Calculate the probability $P\,(0 \le Y \le 1/3)$.

(c) Calculate the probability $P\left(X^2 + Y^2 \le 1\right)$.

(12 marks)

(ii) Let $\omega = (3y^2 + 2y)\,dx + (6xy + 2x + 2y^2)\,dy$.

(a) Show, *without* finding a potential function, that ω is an exact differential.

(b) Now find a potential function f for ω.

(c) Evaluate the line integral $\int_\gamma \omega$, where $\gamma : x = \cos^3 t, y = \sin^3 t,\ \ 0 \le t \le \pi/2$. ***(13 marks)***

2 (i) State Green's Theorem, being careful to include any conditions needed for its validity. Hence evaluate

$$\int_C (xy^2 + ye^{xy})dx + (2x^2y + xe^{xy})dy,$$

where C is the triangular path with vertices $(0,0)$, $(1,0)$ and $(0,1)$, described in the anticlockwise direction. ***(13 marks)***

(ii) Let C be a positively oriented curve in the (x,y)-plane, and let D be the region inside C. Prove, using Green's Theorem, that the area of D is equal to

$$\int_C \frac{3}{4}\,x\,dy - \frac{1}{4}\,y\,dx\ .$$

(4 marks)

(iii) Let C be the curve parametrised by

$$x = \cos^3 t, y = \sin t,\ \ 0 \leq t \leq 2\pi.$$

Using the result from part *(ii)*, or otherwise, calculate the area of the region D enclosed by C. ***(8 marks)***

3 (i) Let

$$F(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n \cos(nx) \;+\; b_n \sin(nx)\right) \;.$$

Prove that

$$\int_{-\pi}^{\pi} F(x) \sin(mx)\, dx \;=\; \pi b_m$$

where m is any positive integer. Evaluate also $\int_{-\pi}^{\pi} F(x) dx$ and $\int_{-\pi}^{\pi} F(x) \cos(mx) dx$.

[You may assume without proof that for positive integers m and n

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx)\, dx \;=\; \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \;, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx)\, dx \;=\; \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \;, \end{cases}$$

and $\int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx = 0$.] ***(8 marks)***

(ii) Let f be the periodic function with period 2π such that

$$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < 0; \\ \pi & \text{for } 0 \leq x < \pi. \end{cases}$$

Show that the Fourier series for f is

$$\frac{\pi}{2} \;+\; 2 \left(\sin(x)) \;+\; \frac{\sin(3x)}{3} \;+\; \frac{\sin(5x)}{5} \;+\; \ldots \right) \;.$$

(12 marks)

(iii) Deduce Gregory's series:

$$\left(1 \;-\; \frac{1}{3} \;+\; \frac{1}{5} \;-\; \ldots\right) \;=\; \frac{\pi}{4} \;.$$

(5 marks)

4 (i) Let $x > 0$. Show that

$$\int_0^{x/2} \frac{1}{\sqrt{x^2 - t^2}}\, dt \;=\; \frac{\pi}{6}\;.$$

By differentiating the above, evaluate

$$\int_0^1 \frac{1}{(4 - t^2)^{3/2}}\, dt\;.$$

(10 marks)

(ii) Recall the Fourier transform $\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-ist}\, dt$ (when it exists), and the Fourier inversion formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{ist}\, ds$$

(valid under certain conditions). Let

$$u(t) = \begin{cases} 1 & \text{if } -1 \le \text{t} \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $\hat{u}(s) = 2\sin(s)/s$.

(b) Evaluate the integral $\int_{-\infty}^{\infty} \sin(x)/x\; dx$. [Hint: use Fourier inversion.]

Recall the convolution $(g * h)(x) := \int_{-\infty}^{\infty} g(t)h(x - t)\, dt$. You may assume the Convolution Theorem: $\widehat{g * h}(s) = \hat{g}(s)\hat{h}(s)$.

(c) Calculate directly $(u * u)(0)$.

(d) Evaluate the integral $\int_{-\infty}^{\infty} (\sin^2 x)/x^2\; dx$. ***(15 marks)***

5 Let $f(x, y) = y^3 + x^2 - 2y^2 + 2xy - 4x - y + 5$.

(i) Find the unique critical point of function f, being careful to show that there are not any more. What happens when you apply the usual test to attempt to classify this critical point? ***(9 marks)***

(ii) Find the Taylor polynomial of degree three for f centred at the critical point. ***(9 marks)***

(iii) What does the Taylor polynomial of degree three for f become (in terms of y) when you restrict to the line $x = 2 - y$ through the critical point? Hence classify the critical point of f. ***(5 marks)***

(iv) What would be the Taylor polynomial of degree ten for f centred at the critical point? ***(2 marks)***

End of Question Paper