The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2009-2010**

NUMERICAL LINEAR ALGEBRA **2 hours**

Answer* four *questions. You are advised* not *to answer more than four questions: if you do, only your best four will be counted.

1. a) Given a set of data $(x_j, f_j),\ j = 0, \ldots, m$ and the basis functions $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$, show that the **normal equations** for the coefficients $\alpha_i,\ i = 0, 1, \ldots, n$ of the least-squares approximation

$$y(x) = \sum_{i=0}^{n} a_i \phi_i(x)$$

to the data are given

$$\sum_{i=0}^{n} \alpha_i \sum_{j=0}^{m} \phi_i(x_j)\phi_k(x_j) = \sum_{j=0}^{m} f_j \phi_k(x_j),$$

for $k = 0, 1, \ldots, n$. ***(6 marks)***

b) Hence, determine the best least-squares fit of the form

$$y(x) = \alpha_0 + \alpha_1 e^x$$

to the data:

x:	0	0.2	0.4	0.5
f:	1.5000	1.6107	1.7459	1.8244

Quote your results correct to five decimal places. ***(14 marks)***

c) Describe briefly how you could use the least-squares technique to obtain an approximation to a set of data using the function

$$p(x) = a 10^{bx} 100^{cx^2}.$$

(5 marks)

2. a) The condition number $\mathcal{K}_p(A)$ of the matrix A in the p-norm is given by the relation

$$\mathcal{K}_p(A) = \|A\|_p \left\|A^{-1}\right\|_p.$$

Verify that the change $\delta\mathbf{x}$ in the solution $\mathbf{x}$ of the system of equations

$$A\mathbf{x} = \mathbf{b}, \qquad (1)$$

induced by a change $\delta\mathbf{b}$ in the right-hand side vector $\mathbf{b}$ satisfies, in the usual notation,

$$\frac{\|\delta\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \le \mathcal{K}_p(A)\frac{\|\delta\mathbf{b}\|_p}{\|\mathbf{b}\|_p}.$$

(6 marks)

b) The residual vector $\mathbf{r}$ is defined by

$$\mathbf{r} = A\hat{\mathbf{x}} - \mathbf{b},$$

where $\hat{\mathbf{x}}$ is a numerical solution of the system of equations, in (1) above, for which $\mathbf{x}$ is an exact solution. Verify that

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_p}{\|\mathbf{x}\|_p} \le \mathcal{K}_p(A)\frac{\|\mathbf{r}\|_p}{\|\mathbf{b}\|_p}.$$

(5 marks)

c) Hence, determine a bound on the relative error in the numerical solution

$$\hat{\mathbf{x}} = (0.0909, -0.1988, 0.3111)^T$$

to the system of equations

$$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0,3 \\ -0.3 \\ 1.1 \end{pmatrix}$$

in the 2-norm. Work to four decimal places. ***(14 marks)***

3. a) The real symmetric matrix A has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ satisfying

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_n|$$

with corresponding linearly independent eigenvectors $\mathbf{x}_i,\ i = 1, 2, ..., n$ which can be supposed normalized so that the largest element of each one is unity. The Power Method for estimating the dominant eigenvalue of the matrix A, given an estimate $\mathbf{z}_0$ of the corresponding eigenvector, takes the form of the iteration:

$$\begin{aligned} \mathbf{y}_k &= A\mathbf{z}_{k-1}, \\ \mathbf{z}_k &= \frac{\mathbf{y}_k}{\mu_k}, \ k = 1, 2, ..., \end{aligned}$$

where μ_k is the element of $\mathbf{y}_k$ of maximum modulus.
Prove that the iteration converges, stating any assumptions made.

(12 marks)

b) The LU factors of the matrix

$$A = \begin{pmatrix} 9 & 6 & 3 \\ 6 & 8 & 4 \\ 3 & 4 & 3 \end{pmatrix}$$

are

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/2 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 9 & 6 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write down the **Inverse** Power Method in terms of L and U to determine estimates of the eigenvalue of A of least magnitude and its eigenvector.
Hence, using

$$\mathbf{z}_0 = (0, 0, 1)^T$$

as initial estimate, perform two iterations of the **Inverse** Power Method to determine estimates of the eigenvalue of A of least magnitude and its eigenvector. Quote your results correct to four decimal places. ***(13 marks)***

4. a) Derive a sufficient condition in terms of norm such that the iteration

$$\mathbf{x}^{(k+1)} = H\mathbf{x}^{(k)} + \mathbf{d} \tag{2}$$

gives a convergent sequence of iterates where H is an iteration matrix and $\mathbf{d}$ is the corresponding vector. ***(8 marks)***

b) Perform two iterations of the Jacobi method on the system of equations

$$\begin{pmatrix} 10 & 2 & 4 \\ 2 & 10 & 1 \\ 4 & 1 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 21 \\ -4 \end{pmatrix},$$

using

$$\mathbf{x} = (1, 1, 1)^T$$

as starting vector. ***(10 marks)***

c) Describe how the dominant eigenvalue of the matrix H, in (2) above, could be estimated as the iteration progresses. ***(7 marks)***

5. a) Given the overdetermined system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

where A is an $m \times n$ matrix, $\mathbf{x}$ is an n-vector and $\mathbf{b}$ is an m-vector, prove that the weighted least-squares solution of the system satisfies the relation

$$A^T W A\mathbf{x} = A^T W\mathbf{b},$$

where W is an $m \times m$ diagonal matrix of weights w_j, $j = 1, 2, ...m.$ ***(8 marks)***

b) Hence write down, but do not solve, the normal equations for a least-squares fit by a polynomial of degree one to the data

x_j:	-0.50	-0.25	0.00	0.25	0.50
b_j:	1.5431	1.1276	1.0000	1.1276	1.5431
w_j:	1	1	1	20	1

(10 marks)

c) Find the Lagrange polynomial of least degree which passes through the points:

$$(x, y) = (-1, 6),\ (0, -2),\ (2, 12).$$

(7 marks)

End Of Question Paper