Data provided: "Statistics Tables" by H.R. Neave

MAS205

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2009–10**

Statistics Core **2 hours**

Marks will be awarded for your best **three** *answers.*
RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 99 marks available on the paper.

1 (a) Let X be a random variable with a chi-squared distribution with ν degrees of freedom, with probability density function

$$f(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2} & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and let $Y = \sqrt{X}$. Find the probability density function of Y. ***(8 marks)***

(b) Let U be a random variable with a uniform distribution on $[-1, 1]$, and let $V = U^2$. Find the distribution function of V, and hence show that the probability density function of V is

$$f_V(v) = \begin{cases} \dfrac{1}{2\sqrt{v}} & 0 \leq v \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(13 marks)

(c) A single observation from a Poisson distribution with unknown mean $\lambda > 0$ gives the value 4.

(i) Write down the likelihood function of λ given this observation. ***(3 marks)***

(ii) Derive the maximum likelihood estimate of λ. ***(9 marks)***

2 Let $T \subseteq \mathbb{R}^2$ be the set $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$, and let $f_{X,Y}$ be the probability density function

$$f_{X,Y}(x,y) = \begin{cases} cxy & (x,y) \in T \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c. ***(6 marks)***

(b) Find

$$P\left(X \geq 2Y\right).$$

(7 marks)

(c) Find the marginal probability density functions of X and Y. ***(8 marks)***

(d) Find the conditional probability density function of Y given $X = x$, and express the conditional expectation of Y given X as a function of X. ***(8 marks)***

(e) Are X and Y independent? Give a reason for your answer. ***(4 marks)***

3 (a) Let X_1, X_2 and X_3 have a multivariate normal distribution with mean vector

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

and covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 9 & -1 \\ 1 & -1 & 16 \end{pmatrix}.$$

(i) Describe the distributions of X_1, X_2 and X_3 in the form $N(\mu, \sigma^2)$, and find the correlation coefficients between X_1 and X_2, between X_1 and X_3, and between X_2 and X_3. ***(6 marks)***

(ii) Define $Y_1 = 5X_1 + X_2$ and $Y_2 = X_3 - X_2$. Find the mean vector and covariance matrix of Y_1 and Y_2, and hence describe their distributions in the form $N(\mu, \sigma^2)$. State whether Y_1 and Y_2 are independent, giving a reason for your answer. ***(13 marks)***

(b) Let $S \subseteq \mathbb{R}^2$ be the set $\{(x, y) : x \geq 0, -x \leq y \leq x\}$, and let

$$f_{X,Y}(x, y) = \begin{cases} 4e^{-(3x+y)} & (x, y) \in S \\ 0 & \text{otherwise} \end{cases}$$

be the joint probability density function of a random vector (X, Y). Let $U = \dfrac{X+Y}{2}$ and $V = \dfrac{X-Y}{2}$.

(i) Find the joint probability density function of U and V. ***(10 marks)***

(ii) Show that U and V are independent, and write down the marginal probability density functions of U and V. ***(4 marks)***

4 (a) A coin has probability θ of showing a head on each toss. It is known that the set Θ of possible values of θ is $\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$. A sequence of five tosses of the coin gives four heads.

(i) Find the likelihood of each of the three possible values of θ.

(5 marks)

(ii) What is the maximum likelihood estimate of θ? ***(3 marks)***

(b) (i) Show that the form of the probability density function of a Gamma distribution $Ga(2, \beta)$ can be reduced to

$$f(x|\beta) = \beta^2 x e^{-\beta x},$$

and show that if X has this distribution then $E\left(\frac{1}{X}\right) = \beta$.

(5 marks)

(ii) Let $x_1, x_2, \ldots, x_n$ be a random sample from a Gamma distribution $Ga(2, \beta)$, where β is unknown. Show that the log likelihood of β is

$$\ell(\beta) = 2n \log \beta + S_1 - \beta S_2,$$

where $S_1 = \sum_{i=1}^{n} \log x_i$ and $S_2 = \sum_{i=1}^{n} x_i$, and hence find the maximum likelihood estimate of β based on the sample. ***(16 marks)***

(iii) In (ii), if it is known that $\beta \geq 2$ and the sample satisfies $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 1.43$, what is the maximum likelihood estimate of β?

(4 marks)

End of Question Paper