The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2009–2010

Topics in Number Theory

2 hours

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

No credit will be given for solutions which rely solely on the use of a calculator. Your solutions should give enough details to make it clear how you arrived at the answer.

1 (i) You publish $(n, e) = (221, 7)$ in the RSA directory and receive 5. Decode it. *(12 marks)*

(ii) Let p be a prime number greater than 5. Show that $10^{p-1} \equiv 1 \pmod{p}$ and deduce that p has a multiple of the form $11\ldots1$. *(5 marks)*

(iii) Determine the remainder when 19! is divided by 69. *(8 marks)*

2 (i) Calculate the Legendre symbols $\left(\frac{424}{17}\right)$ and $\left(\frac{424}{19}\right)$, and deduce that the congruence

$$x^2 + 20x - 6 \equiv 0 \pmod{323}$$

has a solution. Find all its solutions. *(18 marks)*

(ii) Determine for which primes $p > 2$ the congruence

$$x^4 - x^2 - 2 \equiv 0 \pmod{p}$$

has a solution. *(7 marks)*

3 (i) For each of the numbers

$$2^{129} - 1, \qquad 2^{130} + 1,$$

find a prime number which divides it. *(6 marks)*

(ii) State *Euler's Criterion*, and use it to find a prime number which divides $130^{128} + 1$. *(9 marks)*

(iii) Determine the sum of the positive divisors of $14 \times 128 \times 130$. *(5 marks)*

(iv) Show that, for a positive integer n, $2^{2n} \equiv 1 \pmod 3$. Hence or otherwise show that every even perfect number other than 6 is congruent to 1 (mod 9). *(5 marks)*

4 The positive integers x, y, z form a Pythagorean triple and k is the highest common factor of x, y, z. Express x, y, z in terms of three parameters k, s, t. *(2 marks)*

(i) Find the values of k, s, t for the Pythagorean triple

$$10^2 - 1, \quad 13^2 - 1, \quad 14^2 - 1.$$ *(8 marks)*

(ii) Show that there is no primitive Pythagorean triple in which one of the numbers is 2010. *(3 marks)*

(iii) Find all Pythagorean triples x, y, z in which one of the numbers is 2010 and the highest common factor of x, y, z is 2. *(12 marks)*

5 (i) Establish the result

$$\sqrt{21} = [4; \overline{1, 1, 2, 1, 1, 8}],$$

find a convergent of $\sqrt{21}$ which differs from it by less than 10^{-4} and find a solution of Pell's equation

$$x^2 - 21y^2 = 1$$

in positive integers. *(17 marks)*

(ii) Use *Binet's Formula* to prove that

$$f_{2n+2}f_{2n-1} - f_{2n}f_{2n+1} = 1$$

for all $n \geq 1$, where (f_n) denotes the Fibonacci sequence. *(8 marks)*

End of Question Paper