Data Provided: Formula Sheet

The University Of Sheffield.

MAS242

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2009-2010

Mathematics III(Electrical)

2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 z is the complex number $x + jy$.

(i) Sketch the regions in the z-plane corresponding to $x \leq 1$, $y \geq x - 2$, $|z| \leq 2$ and $|z - 4| \leq 1$. ***(8 marks)***

(ii) For the mapping $w = \frac{1}{2}(1 - j)z + j - 1$ where $w = u + jv$,

(a) find $u(x, y)$ and $v(x, y)$;

(b) find the image in the w-plane of the region $x \geq 0$ in the z-plane;

(c) sketch your results in the z- and w-planes. ***(13 marks)***

(iii) Find out if the image of the line $y + 2x = 1$ under the bilinear mapping $w = \dfrac{3z + 2}{z + 1 + 2j}$ is a circle or a straight line. ***(4 marks)***

2 Expand the function $f(z) = \dfrac{3z + j}{(z - j)(z + 2j)}$ into partial fractions and hence

(i) find the first three non-zero terms of the power series expansion of $f(z)$ about the point $z = 0$ using either the Taylor series or the binomial expansion method, Show the region of convergence of the power series and all poles and zeros of $f(z)$ on the Argand diagram. ***(16 marks)***

(ii) Find the first four terms of the Laurent series expansion of $f(z)$ about the point $z = j$. ***(9 marks)***

3 (i) Find all the poles of $f(z) = \dfrac{1}{z(z^2+4)(z+4j)}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)dz$, writing your solutions in the form $a + jb$, where a and b are real, where

(a) C is the circle $|z| = 3$

(b) C is the circle $|z+2| = 1$.

(16 marks)

(ii) By making the substitution $z = e^{j\theta}$, use the method of residues to evaluate the real integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$$

You are given that $\sin\theta = \dfrac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$. ***(9 marks)***

4 (i) The function $x(t)$ satisfies the differential equation

$$\ddot{x} - 2\dot{x} + 5x = \cos t$$

(where dot denotes differentiation with respect to t) and the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. Show that the Laplace transform, $X(s)$, of $x(t)$ is given by

$$X(s) = \frac{1}{10}\left(\frac{2s-1}{s^2+1}\right) - \frac{1}{10}\left(\frac{2s-15}{(s-1)^2+4}\right)$$

Hence determine $x(t)$ for $t > 0$. ***(15 marks)***

(ii) The unit pulse function, $\Pi(t)$, is defined by

$$\Pi(t) = 1 \text{ for } |t| \leq \frac{1}{2} \text{ and } \Pi(t) = 0 \text{ for } |t| > \frac{1}{2}.$$

Sketch $\Pi(t)$ and $\Pi(2t)$. Using direct integration, show that the Fourier transform of $\Pi(t)$ is $\text{sinc}\left(\dfrac{\omega}{2}\right)$. Sketch $\text{sinc}\left(\dfrac{\omega}{2}\right)$ and, using the scaling property of the Fourier transform, sketch the Fourier transform of $\Pi(2t)$.

(10 marks)

End of Question Paper

FORMULA SHEET

Table of Laplace Transforms

$f(t)$	$F(s)$	Region of validity
constant $= c$	$\frac{c}{s}$	$Re(s) > 0$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$Re(s) > \alpha$
t	$\frac{1}{s^2}$	$Re(s) > 0$
$\cos kt$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$\sin kt$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
t^n	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$t^n e^{\alpha t}$	$\frac{n!}{(s-\alpha)^{n+1}}$	$Re(s) > \alpha$
$e^{\alpha t}\sin kt$	$\frac{k}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$e^{\alpha t}\cos kt$	$\frac{s-\alpha}{(s-\alpha)^2+k^2}$	$Re(s) > \alpha$
$\delta(t-T)$	e^{-sT}	delta function
$H(t-T)$	$\frac{e^{-sT}}{s}$	step function
$H(t) - H(t-T)$	$\frac{1}{s}(1-e^{-sT})$	rectangular pulse

Note: in this table the parameters α and k are real constants and H is the Heaviside step function.

Some general properties of the Laplace transform

In the following table the notation $\mathbf{L}\{f(t)\} = F(s)$ has been used.

$\mathbf{L}\{af(t) + bg(t)\} = a\mathbf{L}\{f(t)\} + b\mathbf{L}\{g(t)\}$	linearity
$\mathbf{L}\{\frac{d}{dt}f(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathbf{L}\{\frac{d^2}{dt^2}f(t)\} = s^2F(s) - sf(0) - f'(0)$	differentiation twice with respect to t
If $g(t) = \int_0^t f(u)du$ then $\mathbf{L}\{g(t)\} = \frac{1}{s}F(s)$	integration
$\mathbf{L}\{tf(t)\} = -\frac{dF}{ds}$	differentiation w.r.t. s
$\mathbf{L}\{e^{-kt}f(t)\} = F(k+s)$	shift
$\mathbf{L}\{f(at)\} = \frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	scaling
$\mathbf{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$	time delay

Convolution

For causal functions

$$f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

and has Laplace transform $F(s)G(s)$.

Fourier transform

The Fourier transform $\mathbf{F}(\omega)$ of a function $f(t)$ is defined by

$$\mathbf{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt.$$

The time shift property: the Fourier transform of a function $f(t-T) = e^{-j\omega T}\ \mathbf{F}(\omega)$.

The scaling property: the Fourier transform of a function $f(at) = \frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$.

Residues

The general formula for the residue at a pole, z_0, of order m is

$$\frac{1}{(m-1)!}lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right).$$