The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2009-2010**

**Mathematics III (Control)** **2 hours**

*Answer* ***fou****r questions. If you answer more than four questions, only your best four will be counted.*

**1.** (a) Find the line of intersection of the planes

$$2x - 3y + 5z = 7 \text{ and } 5x - 8y + 7z = 10,$$

giving the equation in parametric form. (9 marks)

(b) Find the Cartesian equation of the plane which contains the points

$$(2,5,-1), (3,1,7), (-1,4,-6).$$

(8 marks)

(c) Find the Hermite form of the matrix

$$\begin{pmatrix} 0 & 1 & -3 & 2 & -5 \\ 0 & -2 & 6 & -4 & 9 \\ 0 & 3 & -9 & 6 & -10 \\ 0 & -4 & 17 & -18 & 13 \end{pmatrix},$$

specifying each row operation using any appropriate notation. (8 marks)

**2.** (i) Show that $(3,9,-13), (3,10,18), (2,6,-9)$ is a basis for $\mathbb{R}^3$ and that one of $(5,-2), (-10,4)$ and $(3,4), (4,5)$ is a basis for $\mathbb{R}^2$, but the other is not. (8 marks)

(ii) Let $T$ be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by the formula $T(x, y, z) = (x + 2y - 3z,\ 2x + 5y - 2z)$. Find the matrix of $T$ with respect to the bases given in part (i) above. (8 marks)

(iii) Find bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ so that the linear transformation $T$ of part (ii) above is in Smith form. (9 marks)

**3.** A real $8\times8$ matrix $A$ has minimum polynomial $x^4-2x^3+x^2-12x+20$.
The trace of $A$ is $4$.
The geometric multiplicity of the eigenvalue $2$ is $3$.

(i) Find all the eigenvalues of $A$ (5 marks)

(ii) Find the algebraic multiplicity of the complex eigenvalues. (5 marks)

(iii) Write down the invariant factors of $xI-A$. (5 marks)

(iv) Write down the rational form of $A$. (5 marks)

(v) Write down the Jordan form of $A$. (5 marks)

**4.** Consider the second order differential equation $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\dfrac{\mathrm{d}x}{\mathrm{d}t}+10x=0$.

(i) By introducing a new variable $y$, write this equation in the form $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=A\mathbf{x}$ where $\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}$ and $A$ is a matrix. (5 marks)

(ii) Find the eigenvalues and eigenvectors of $A$. (The eigenvectors should be scaled so that the first coordinate in each case is equal to 1.) (8 marks)

(iii) Hence find a matrix $P$ such that $P^{-1}AP=J$, the Jordan form of $A$. (4 marks)

(iv) Hence solve the differential equation, using the formula $\mathrm{e}^{\pm\mathrm{j}q}=\cos q\pm\mathrm{j}\sin q$ to write the general solution in the form

$$x=B\mathrm{e}^{-kt}\cos pt+C\mathrm{e}^{-kt}\sin pt$$

where the values of $k$ and $p$ are to be found. (8 marks)

**5.** (i) Give a matrix representation of the quadratic form

$$Q(x,y,z)=15x^2+29y^2+22z^2+12xz+12yz$$

(5 marks)

(ii) Show that $\begin{pmatrix}2\\9\\6\end{pmatrix}$ is an eigenvector of the matrix and find a normalised eigenvector with the same eigenvalue. (5 marks)

(iii) Find a normalised eigenvector associated with the eigenvalue 22. (5 marks)

(iv) By taking the cross product of the eigenvectors in parts (ii) and (iii), or otherwise, find a third eigenvalue and normalised eigenvector of the matrix. (5 marks)

(v) Hence write $Q(x,y,z)$ as a sum or difference of squares and describe the surface $Q(x,y,z)=11$. (5 marks)

**6.** Let $z=\left(2x^2+3y^2\right)e^{-x^2-y^2}$ .

(i) Show that there are $5$ critical points of $z$ in the $(x,y)$- plane, and find their coordinates. (11 marks)

(ii) Find the values of $z$ at the critical points. (3 marks)

(iii) Using the Hessian matrix classify each critical point as a local maximum, a local minimum or a saddle point. (11 marks)

## End of Question Paper