The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2009–10

Further Civil Engineering Mathematics and Computing — **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Use L'Hopital's rule to determine the limit

$$\lim_{x \to 0} \frac{x - \sin x}{x^2}$$

(5 marks)

(ii) Give the general form of the Taylor expansion of a function $f(x)$ about a point $x = a$. Expand the function $f(x) = tan^{-1}(x/2)$ in Taylor series about $x = 2$ up to the term proportional to x^3. Hint

$$\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{x^2 + a^2}$$

(10 marks)

(iii) Using the Newton-Raphson method, given as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots,$$

and assuming an initial guess $x_0 = 0.8$, calculate the root of the equation $f(x) = \ln(x+1) - 8x + 3 = 0$ correct to three decimal places. ***(6 marks)***

(iv) Using Descartes' method of finding roots of a polynomial with real coefficients, find the maximum number of real positive and negative roots of the polynomial

$$f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$$

(4 marks)

2 (i) Show that the equation

$$x^2 - e^x - 6 = 0$$

has a root in the interval $(-2.6, -2.2)$ and perform *five* iterations of the bisection method to obtain a refined estimate of the interval which contains the root. Work correct to three decimal places. ***(10 marks)***

(ii) The function $y(x)$ satisfies the ordinary differential equation

$$\frac{dy}{dx} = x^2 + \sin y + 1$$

and the initial condition $y(0) = 0$. Determine the Taylor series solution of this equation as far as the term in x^3. Use this series to evaluate $y(0.2)$ giving your answer correct to four decimal places. ***(8 marks)***

(iii) Values of y at $x = 2$ determined using the fourth-order Runge-Kutta method with two different step-lengths are given in the following table

h	y(2)
0.2	3.40978
0.4	3.39278

Use this data to estimate a value for h which will ensure that the error in the calculated value of $y(2)$ using a fourth-order Runge-Kutta method does not exceed 10^{-4}. The local error of the Runge-Kutta method is given by

$$Y(x_n) - y(x_n) \approx Ch^4$$

where $Y(x_n)$ is the exact solution, $y(x_n)$ is the numerical solution and C is a constant. Give your answer correct to 4 decimal places. ***(7 marks)***

3 (i) Verify that

$$V = \frac{1}{\sqrt{t}} e^{-x^2/4Kt}$$

is a solution of the heat conduction equation

$$\frac{\partial V}{\partial t} = K \frac{\partial^2 V}{\partial x^2}$$

(6 marks)

(ii) The length, L, of the diagonal of a rectangular tank of sides x, y, and z is to be calculated as

$$L = \sqrt{x^2 + y^2 + z^2}$$

where x, y and z are measured to be 1 m, 2 m and 2 m, respectively. Each of these measurements are accurate to ± 0.03 m. Estimate the maximum resulting error in L. ***(9 marks)***

(iii) If $z = e^{xy^2}$, $x = t\cos t$ and $y = t\sin t$, use the chain rule to calculate dz/dt when $t = \pi/2$. ***(10 marks)***

4 (i) Show that the Fourier series expansion of the function $f(x) = e^x$ obtained in the interval $-\pi \le x \le \pi$ is

$$e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)\right]$$

(18 marks)

Hint

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(ii) An explicit approximation to the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad 0 < x < 1$$

is given (in the usual notations) by

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}$$

where $r = k/h^2$. If the initial and boundary conditions associated with the heat conduction equation are given by

$$u(x,0) = 2x^2, \quad 0 < x < 1$$

$$u(0,t) = 0, \quad \text{and} \quad u(1,t) = 2, \quad t > 0$$

use the above explicit scheme, with $h = 0.2$ and $r = 0.5$ to calculate grid-points values of u at the first time step. Write down the value of t at the first time-step. Work correct to 3 decimal places. ***(7 marks)***

5 (i) The variation of heat in a rod of length L is given by the equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} - u, \qquad u = u(x,t) \tag{1}$$

By using the method of separation of variables show that

$$\frac{T'}{2T} + \frac{1}{2} = \frac{X''}{X} = \alpha$$

where α is a constant. ***(5 marks)***

Assuming that $\alpha = -s^2$, $(s \neq 0)$ show that there is a solution of the equation (1) of the general form

$$u(x,t) = (A\cos sx + B\sin sx)e^{-(1+2s^2)t}$$

and show that the corresponding solution when $s = 0$ is

$$u(x,t) = (bx + c)e^{-t}$$

(11 marks)

 Question 5 continued on next page

5 (continued)

Given that for $t \geq 0$ $\partial u/\partial x = 0$ at $x = 0$ and $x = L$, show that the solution of equation (1) which satisfies these boundary conditions consists of a linear combination of terms of the form

$$a_n \cos \frac{n\pi x}{L} e^{-(1+2n^2\pi^2/L^2)t}$$

where n is an integer and the a_n are arbitrary constants. ***(9 marks)***

End of Question Paper