The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2009-2010

Vectors and Fluids

2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 The vector field $\mathbf{u}$, where

$$\mathbf{u} = (4yz, -4yz + Azx, 4xy - 2y^2 + Bz^2),$$

is both solenoidal ($\nabla \cdot \mathbf{u} = 0$ everywhere) and irrotational ($\nabla \times \mathbf{u} = \mathbf{0}$ everywhere). Given that A and B are constants, show that $A = 4$ and $B = 2$. In the remainder of the question, use these values of A and B. *(7 marks)*

Determine a scalar potential ϕ for $\mathbf{u}$ (i.e. ϕ such that $\mathbf{u} = \nabla\phi$). *(9 marks)*

Verify that

$$\mathbf{A} = (-2yz^2 + 2z^2x + \tfrac{2}{3}y^3 - 2xy^2, 0, 2y^2z),$$

is a vector potential for $\mathbf{u}$ (i.e. $\mathbf{A}$ satisfies $\nabla \times \mathbf{A} = \mathbf{u}$). Show further that

$$\nabla(\nabla \cdot \mathbf{A}) = \nabla^2\mathbf{A}.$$

(9 marks)

2 (i) Use suffix notation and the formula

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

to show that

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u}.\mathbf{u}) - (\mathbf{u}.\nabla)\mathbf{u}.$$

(You are reminded that the k component of $\nabla \times \mathbf{u}$ is $\epsilon_{klm}\dfrac{\partial u_m}{\partial x_l}$, and you are advised to begin by explaining why the i component of $\mathbf{u} \times (\nabla \times \mathbf{u})$ is $\epsilon_{ijk}\epsilon_{lmk}u_j\dfrac{\partial u_m}{\partial x_l}$.) *(9 marks)*

(ii) By considering LL^T and $\det L$, find the unique values of the constants α and β for which

$$L = \alpha\begin{pmatrix} 1 & \sqrt{3} & \beta \\ \beta & 0 & -2 \\ -\sqrt{3} & 1 & 0 \end{pmatrix},$$

is a transformation matrix for the rotation of axes from an undashed right-handed frame $Ox_1x_2x_3$ to a dashed right-handed frame $Ox_1'x_2'x_3'$.

(9 marks)

Find the components in the dashed frame of the vector with components (a, b, c) in the undashed frame. Hence, or otherwise, find the axis about which L represents a rotation. *(7 marks)*

3 Curvilinear coordinates $(\alpha_1, \alpha_2, \alpha_3)$ with $0 \leq \alpha_1 < \infty$, $-\infty < \alpha_2 < \infty$, $-\infty < \alpha_3 < \infty$ are related as follows to the Cartesian coordinates (x, y, z), where k is a positive constant:

$$x = \frac{k}{2}(\alpha_1^2 - \alpha_2^2), \quad y = k\alpha_1\alpha_2, \quad z = \alpha_3.$$

Show that, in the usual notation,

$$\delta x \approx k(\alpha_1 \delta\alpha_1 - \alpha_2 \delta\alpha_2),$$

and find the analogous expressions for δy and δz.
Use your results to find an expression for

$$(\delta \mathbf{r}).(\delta \mathbf{r}) = (\delta s)^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$$

in terms of $k, \alpha_1, \alpha_2, \delta\alpha_1, \delta\alpha_2$ and $\delta\alpha_3$, where $\delta\mathbf{r} = (\delta x, \delta y, \delta z)$. Explain why it can now be asserted that $(\alpha_1, \alpha_2, \alpha_3)$ are orthogonal coordinates, and write down expressions for h_1, h_2, h_3 (in the usual notation) in terms of k, α_1 and α_2.
(10 marks)

Consider the coordinate surface $\alpha_3 = 0$ (which is, of course, the xy plane). Show that the equation of the curve in which the coordinate surface $\alpha_1 = k^{-\frac{1}{2}}$ meets the xy plane is

$$y^2 = 1 - 2x.$$

Find also the equation of the curve in which the coordinate surface $\alpha_2 = 2k^{-\frac{1}{2}}$ meets the xy plane. Draw a sketch showing these two curves. Find the coordinates of their points of intersection, and state, without further calculation but giving a reason, the angle at which the curves cross at these points. *(15 marks)*

4 The velocity field in the steady two-dimensional flow of an incompressible fluid of uniform density ρ is given by

$$\mathbf{u} = U\{\cos(\frac{\pi x}{a})\cos(\frac{\pi y}{a})\mathbf{i} + \sin(\frac{\pi x}{a})\sin(\frac{\pi y}{a})\mathbf{j}\},$$

where U and a are constants.

(i) Show that $\nabla \cdot \mathbf{u} = 0$ and find the stream function $\psi(x, y)$ such that $\mathbf{u} = \nabla \times \psi(x, y)\mathbf{k}$.
(7 marks)

(ii) Find the acceleration of the fluid which is $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in this case.
(10 marks)

(iii) In this flow gravity can be ignored. Use Euler's equation to show that the pressure p satisfies

$$p = \frac{1}{4}\rho U^2\{\cos(\frac{2\pi y}{a}) - \cos(\frac{2\pi x}{a})\} + p_0,$$

where p_0 is a constant. *(8 marks)*

5 In spherical polar coordinates (r, θ, ϕ) with $r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi$, the velocity field $\mathbf{u}$ in the irrotational flow of an incompressible fluid of uniform density ρ satisfies

$$\mathbf{u} = \nabla W, \quad W = \frac{\beta \cos \theta}{r^2},$$

where β is a constant. You may assume that $\nabla^2 W = 0$, but state why this condition is necessary. *(2 marks)*

(i) This is the flow outside the solid sphere $r = a$ which is moving with velocity $U\mathbf{k}$, where, as usual, $\mathbf{k}$ is a unit vector along Oz. The fluid is at rest far away. Given that

$$\nabla W = \frac{\partial W}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial W}{\partial \theta}\hat{\boldsymbol{\theta}},$$

show that

$$\beta = -\frac{1}{2}Ua^3.$$

(8 marks)

(ii) The kinetic energy T of the fluid is given by

$$T = \frac{1}{2}\rho \int \mathbf{u} \cdot \mathbf{u} \, dV,$$

where the volume integral is over the region $r \geq a$. Given that $h_1 = 1$, $h_2 = r, \ h_3 = r \sin \theta$ in standard notation, find T. *(15 marks)*

End of Question Paper