The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** — **Spring Semester 2009–2010**

# Rings and Groups

**2 hours**

*Answer **all questions**.*

*Question 1 is multiple choice; there is exactly one correct answer in each case. You will be awarded one mark for each correct answer and no marks for each incorrect answer. You are not required to justify your answers.*

*For questions 2–5 you should justify your answers carefully unless the question states otherwise.*

**1** Multiple choice. You should write down exactly one answer to each of the following twenty questions. **Please write down your answers in a clear list, separate from any rough work.** You are not required to justify your answers.

(i) Which of the following is the multiplicative inverse of 46 in $\mathbb{Z}_{1523}$?

(a) 53

(b) 0.0217

(c) 298

(d) 1477

(ii) Which of the following is a zero-divisor in $\mathbb{Z}_{459}$?

(a) 34

(b) 44

(c) 50

(d) 61

(iii) Which of the following rings is an integral domain?

(a) $\mathbb{Z}_6$

(b) $\mathbf{Mat}_2(\mathbb{R})$

(c) $\mathbb{Z}_6[x]$

(d) $\mathbb{Z}_{37}$

(iv) Which of the following is a field?

(a) $\mathbb{Z}$

(b) $\mathbb{C}$

(c) $\mathbb{Z}[x]$

(d) $\mathbf{Mat}_5(\mathbb{R})$

(v) Which of the following is a subring of $\mathbb{Z}$?

(a) $\mathbb{Z}_2$

(b) The odd numbers

(c) $\mathbb{Z}[x]$

(d) $\mathbb{Z}$

(vi) Which of the following is the best statement of the associativity axiom for multiplication in a ring $R$?

(a) $a(bc) = (ab)c$.

(b) For all $a, b \in R$, $ab = ba$.

(c) For all $a, b, c \in R$, $a(b + c) = ab + ac$.

(d) For all $a, b, c \in R$, $(ab)c = a(bc)$.

**1** (continued)

(vii) What is the negation of the following statement?

$$\forall x \in R \; x.0 = 0$$

(a) $\forall x \in R, \; x.0 \neq 0$

(b) $\forall x \notin R \; x.0 \neq 0$

(c) $\exists x \in R$ s.t. $x.0 \neq 0$

(d) $\exists x \in R$ s.t. $x.0 = x$

(viii) Let $R$ be a commutative ring. Which of the following statements says that $r \in R$ is not a zero-divisor?

(a) There exists $s \in R$ such that $rs = 0$.

(b) For all $s \in R$, $rs = 0 \implies s = 0$.

(c) For all $s \in R$, $rs = 0 \implies r = 0$ or $s = 0$.

(d) $s \neq 0 \implies rs = 0$.

(ix) Which of the following is not a ring axiom?

(a) $\forall a, b, c \in R \;\; (ab)c = a(bc)$

(b) $\exists 1 \in R$ s.t. $\forall a \in R \quad a.1 = a = 1.a$

(c) $\exists 0 \in R$ s.t. $\forall a \in R \quad a.0 = 0 = 0.a$

(d) $\exists 0 \in R$ s.t. $\forall a \in R \quad a + 0 = a = 0 + a$

(x) In the symmetric group $S_5$ how many elements are there in the conjugacy class of the element (12)(34)?

(a) 15

(b) 24

(c) 30

(d) 60

(xi) Let $\alpha$ be an element of the symmetric group $S_9$ with

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 1 & 4 & 3 & 2 & 7 & 8 & 9 \end{pmatrix}.$$

Writing $\alpha$ in disjoint cycle notation gives:

(a) (62)(315)(4789)

(b) (135)(26)

(c) (561432789)

(d) (315)(62)

 Question 1 continued on next page

**1** (continued)

(xii) What is the order of $D_4$, the symmetry group of a square?

(a) 4

(b) 8

(c) 12

(d) 24

(xiii) What is the order of the element $a^6$ in the cyclic group of order 31 generated by $a$?

(a) 6

(b) 24

(c) 30

(d) 31

(xiv) What is the class equation of a cyclic group of order 6?

(a) $1+1+1+1+1+1=6$

(b) $1+1+1+1+2=6$

(c) $1+1+2+2=6$

(d) $1+2+3=6$

(xv) What is the class equation of the alternating group $A_3$?

(a) $1+1+1=3$

(b) $1+2+3=6$

(c) $1+2=3$

(d) $2+2=4$

(xvi) What is the class equation of $D_4$?

(a) $2+2+2+2=8$

(b) $1+3+6+6+8=24$

(c) $1+1+2+2+2=8$

(d) $1+2+4=8$

(xvii) What is the order of the quotient group $S_4/A_4$?

(a) 2

(b) 4

(c) 12

(d) 24

 Question 1 continued on next page

**1** (continued)

(xviii) Let $G$ be a cyclic group of order 24 generated by $g$. Let $H$ be the subgroup generated by $g^4$. What is the order of the quotient group $G/H$?

(a) 2

(b) 4

(c) 6

(d) 24

(xix) Let $\theta$ be a surjective homomorphism $S_4 \longrightarrow S_3$. What is the order of the image of $\theta$?

(a) 1

(b) 4

(c) 6

(d) 24

(xx) Let $\theta$ be an injective homomorphism $S_3 \longrightarrow S_4$. What is the order of the image of $\theta$?

(a) 1

(b) 4

(c) 6

(d) 24

***(20 marks)***

**2** (i) Use Euclid's algorithm to find the multiplicative inverse of 65 in the ring $\mathbb{Z}_{288}$. ***(6 marks)***

(ii) Exhibit a proper zero-divisor in the ring $\mathbb{Z}_{288}$, justifying your answer. ***(3 marks)***

(iii) Let $d$ be a square-free integer with $d \neq 1$. Recall that the norm of an element $r = a + b\sqrt{d}$ of $\mathbb{Z}[\sqrt{d}]$, where $a, b \in \mathbb{Z}$, is given by

$$N(a + b\sqrt{d}) = |a^2 - b^2 d|.$$

Show that $\mathbb{Z}[\sqrt{-5}]$ has no element of norm 3. Hence show that any element of norm 9 or 27 is irreducible. ***(6 marks)***

*Additional marks for rigour and presentation.* ***(5 marks)***

**3** (i) Write down the units in the ring $\mathbb{Z}[i]$, and show that the multiplicative group they form is cyclic. ***(4 marks)***

(ii) Consider the following two factorisations of 13 in $\mathbb{Z}[i]$:

$$13 = (2+3i)(2-3i)$$

$$13 = (3-2i)(3+2i)$$

Do these two factorisations show that $\mathbb{Z}[i]$ is not a unique factorisation domain? ***(4 marks)***

(iii) Give an example of a unique factorisation domain. You do not need to justify your answer. ***(2 marks)***

(iv) Find a unit $r \in \mathbb{Z}[\sqrt{35}]$ with $r > 1$ and **hence** show that the group of units of $\mathbb{Z}[\sqrt{35}]$ is infinite. ***(5 marks)***

*Additional marks for rigour and presentation.* ***(5 marks)***

**4** (i) Write down all possible cycle types in $S_4$, together with the number of elements in $S_4$ of each type. Hence write down the class equation for $S_4$. Justify your answers. ***(11 marks)***

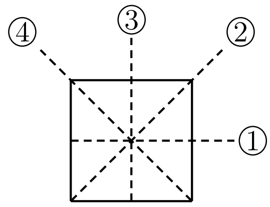
(ii) Is $\langle (123) \rangle$ a normal subgroup of $S_4$? Is $\langle (1234) \rangle$ a normal subgroup of $S_4$? Justify your answers. ***(4 marks)***

*Additional marks for rigour and presentation.* ***(5 marks)***

**5** (i) Let $H$ be a normal subgroup of a group $G$. Describe the quotient group $G/H$. (You need to specify what the elements are, how multiplication on $G/H$ is defined, what the identity is, and what the inverse of a given element is, but you do not need to prove any of your assertions.) ***(5 marks)***

(ii) State, without proof, the First Isomorphism Theorem for groups. ***(2 marks)***

(iii) Consider the square with lines of symmetry labelled 1, 2, 3 and 4 as below.




Recall that $D_4$ is the group of symmetries of the square, and write $e$ for the identity of the group, $a$ for rotation through $\frac{\pi}{2}$ anti-clockwise, and $b_i$ for reflection in the line $i$. Consider the action of $D_4$ on the diagonals of the square.

Find the kernel and image of the homomorphism $f : D_4 \longrightarrow S_2$ induced by the action, justifying your answer. What does the First Isomorphism Theorem tell us in this case? ***(8 marks)***

*Additional marks for rigour and presentation.* ***(5 marks)***

## End of Question Paper