The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2009–2010

**Vector spaces and Fourier theory** — 2 hours

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

**1** (i) Let $V$ be a vector space, and let $\mathcal{V}$ denote the collection of vectors $v_1, \ldots, v_n$ in $V$. Define what it means for $\mathcal{V}$ to

(a) be linearly independent;

(b) span $V$;

(c) be a basis for $V$. ***(6 marks)***

(ii) Explain why none of the following subsets of $\mathbb{R}^3$ are subspaces:

(a) $U_1 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x - y = 1\}$;

(b) $U_2 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x + y \leq z\}$;

(c) $U_3 = \{(x, y, z)^T \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}$. ***(6 marks)***

(iii) If $U$ and $W$ are subspaces of a vector space $V$, show that $U \cap W$ is also a subspace. ***(5 marks)***

(iv) Suppose that $U$ and $W$ are two subspaces of a vector space $V$.

(a) Define $U + W$ and write down a relation between the dimensions of the subspaces $U$, $W$, $U \cap W$ and $U + W$. ***(3 marks)***

(b) Suppose that $\dim V = 7$ and that $\dim U = 5$ and $\dim W = 4$. What are the possible values for $\dim U \cap W$? ***(2 marks)***

(v) In the vector space $V = \mathbb{R}^3$, let $U$ be the subspace $\{(x, y, z)^T \mid x = y = z\}$, and $W$ be the subspace $\{(x, y, z)^T \mid x = 0\}$. Prove that $U \cap W = \{0_V\}$, and $V = U + W$. ***(3 marks)***

**2** (i) In the space $C(\mathbb{R})$ of continuous functions $\mathbb{R} \longrightarrow \mathbb{R}$, show that $f(x) = \sin x$, $g(x) = \cos x$ and $h(x) = x^2$ are linearly independent. ***(4 marks)***

(ii) Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a basis for the vector space $V$. Show that every vector $v \in V$ has a unique expression as a linear combination of vectors in $\mathcal{V}$. ***(4 marks)***

(iii) State the Steinitz Exchange Lemma. ***(2 marks)***

(iv) Let $\mathcal{V} = \{v_1, \ldots, v_m\}$ be a set of vectors in an $n$-dimensional vector space $V$. Say whether or not each of the following statements are necessarily true, and, if not, give a counterexample:

(a) If $\mathcal{V}$ spans, then $m \geq n$;

(b) If $m \geq n$, then $\mathcal{V}$ spans;

(c) If $\mathcal{V}$ is linearly independent, then $m \leq n$;

(d) If $m = n$, then $\mathcal{V}$ is a basis. ***(6 marks)***

(v) (a) Let $V$ be a vector space with basis $\{v_1, v_2, v_3, v_4\}$. Show that the set $\{v_1, v_2 - v_1, v_3 - v_2, v_4 - v_3\}$ is also a basis of $V$. ***(3 marks)***

(b) Consider the vector space $\mathbb{R}[x]_{\leq 3}$ of real polynomials in the variable $x$ of degree at most 3. Recall that we have a standard basis:

$$f_0(x) = 1,\ f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^3,$$

and define $g_0 = f_0,\ g_1 = f_1 - f_0,\ g_2 = f_2 - f_1,\ g_3 = f_3 - f_2$. Let $f(x) = a + bx + cx^2 + dx^3$. Write $f(x) = \alpha g_0 + \beta g_1 + \gamma g_2 + \delta g_3$, and give a matrix $A \in M_4(\mathbb{R})$ such that

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$ ***(4 marks)***

(vi) Give a basis for the vector space

$$V = \{f \in \mathbb{R}[x]_{\leq 3} \mid f''(1) = f(0) = 0\}.$$ ***(2 marks)***

**3** (i) Suppose that $V$ and $W$ are vector spaces over $\mathbb{R}$. What does it mean for a map $\phi : V \longrightarrow W$ to be a *linear map*? ***(3 marks)***

(ii) Define the *kernel* and *image* of a linear map $\phi : V \longrightarrow W$, and prove that the image is a subspace of $W$. ***(6 marks)***

(iii) Consider the subspace $V$ of $C(\mathbb{R})$ spanned by $e^x$, $xe^x$ and $x^2e^x$. (You may assume that these are linearly independent functions.) Define the differentiation map by $D(f) = \dfrac{df}{dx}$. Show that if $f \in V$, then $D(f) \in V$.

Show that the map $D : V \longrightarrow V$ is linear, and give its matrix with respect to the basis $\{e^x, xe^x, x^2e^x\}$.

What are the eigenvalues of $D$? What are the eigenvectors? ***(8 marks)***

(iv) Consider the linear map $\phi : \mathbb{R}[x]_{\leq 3} \longrightarrow \mathbb{R}^4$ given by sending $f \in \mathbb{R}[x]_{\leq 3}$ to $\phi(f) = (f'(0), f(0) - f(1), f(2) - f(3), f'(3))^T$. Find bases of $\mathbb{R}[x]_{\leq 3}$ and $\mathbb{R}^4$ with respect to which the matrix of $\phi$ is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. ***(8 marks)***

**4** (i) Define the notion of an *inner product* on a finite-dimensional vector space over $\mathbb{R}$. ***(5 marks)***

(ii) Let $V$ be an inner product space over $\mathbb{R}$.

(a) What does it mean for a set of vectors $\mathcal{V} = \{v_1, \ldots, v_n\}$ to be orthogonal? ***(1 mark)***

(b) If $\mathcal{V}$ is orthogonal and each $v_i \neq 0$, show that $\mathcal{V}$ is linearly independent. ***(4 marks)***

(iii) Let $V$ be an inner product space (over $\mathbb{R}$). Suppose that $u$ and $v$ are two vectors with $\|u\| = 1$ and $\|v\| = 2$. Show that $2u + v$ and $2u - v$ are orthogonal. ***(3 marks)***

(iv) Consider $\mathbb{R}[x]_{\leq 2}$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) Find an orthogonal basis for this space, using the Gram-Schmidt process on the basis 1, $x$, $x^2$. ***(7 marks)***

(b) What is the nearest linear polynomial to $1 - x + x^2$? ***(5 marks)***

**5** (i) Let $V$ be an inner product space (over $\mathbb{R}$). State the Cauchy-Schwarz inequality for vectors $v, w \in V$, including the criterion for when equality holds.

When $V$ is the inner product space $\mathbb{R}[x]_{\leq 2}$ with the inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx,$$

deduce that

$$\left(\int_0^1 (3x-1)f(x)\,dx\right)^2 \leq \int_0^1 f(x)^2\,dx.$$

Give a function $f$ with $\|f\| = 2$ where this inequality is an equality.

***(9 marks)***

(ii) Consider the Fourier inner product space $C[-\pi, \pi]$ of continuous functions $[-\pi, \pi] \longrightarrow \mathbb{R}$ with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt,$$

(a) Is the set $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \ldots\}$ orthogonal? Is it orthonormal? Justify your answers briefly. ***(2 marks)***

(b) Compute the cosine of the angle between $\sin t \cos 5t$ and $\sin 3t \cos t$.

***(8 marks)***

(c) Consider the subspace $V$ of the Fourier inner product space of trigonometric polynomials of degree at most 1, spanned by the set $\mathcal{V} = \{1, \cos t, \sin t\}$, and the space $M_2(\mathbb{R})$ of $2 \times 2$ real matrices with inner product $\langle A, B \rangle = \operatorname{trace}(A^T B)$. If $v = \alpha + \beta \cos t + \gamma \sin t$, define $\phi(v) = \begin{pmatrix} \alpha & \beta \\ -\beta & \gamma \end{pmatrix}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, compute $\langle \phi(v), A \rangle$, and hence give the adjoint map $\hat{\phi} : M_2(\mathbb{R}) \longrightarrow V$. ***(6 marks)***

## End of Question Paper