The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2009–2010**

Nonlinear Mathematics **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Let $\varphi\colon U \to V$ be a map where U and V are subsets of $\mathbb{R}^2$.

(a) For $S \subseteq U$ define the image $\varphi(S)$ of S under φ.

(b) For $T \subseteq V$ define the preimage $\varphi^{-1}(T)$ of T under φ. **(2 marks)**

(ii) Consider the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(x, y) = (y, xy)$.

(a) Find the image $\varphi(\mathbb{R}^2)$ of φ.

(b) Write $T = \{(u, v) \in \mathbb{R}^2 \mid u = v\}$. Find the preimage $\varphi^{-1}(T)$. **(7 marks)**

(iii) Let $(a, b) \in \mathbb{R}^2$ and let $r > 0$. Define the open ball $B((a, b), r)$. Define what it means for a set $U \subseteq \mathbb{R}^2$ to be open. **(4 marks)**

(iv) Prove directly from your definition in (iii) that the half-plane $H = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ is an open set in $\mathbb{R}^2$. **(6 marks)**

(v) Let U_1 and U_2 be open sets in $\mathbb{R}^2$. Prove that the intersection $U_1 \cap U_2$ is an open set in $\mathbb{R}^2$. **(6 marks)**

2 (i) Let $U \subseteq \mathbb{R}^2$ be an open set and let (a, b) be a point surrounded by U. Let $F\colon U \to \mathbb{R}$ be a function. Define what it means for F to have the limit L at (a, b). **(4 marks)**

(ii) Let $U \subseteq \mathbb{R}^2$ be an open set and let $F\colon U \to \mathbb{R}$ be a function. Let I be a subset of $\mathbb{R}$. State in full (but do not prove) the Theorem which gives conditions under which $F^{-1}(I)$ is an open subset of $\mathbb{R}^2$. **(3 marks)**

(iii) Using (ii) where relevant, decide in each case below whether the set is open or not, proving your assertion.

(a) The set $S_1 = \{(x, y) \in \mathbb{R}^2 \mid \sin x < y < 2 + \sin x\}$.

(b) The set $S_2 = \{(x, y) \in \mathbb{R}^2 \mid y^2 \geqslant 4x\}$.

(c) The set $S_3 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 1\}$. **(13 marks)**

(iv) Let (a, b) be a point of $\mathbb{R}^2$ and let $r > 0$. Show that there are open intervals I and J in $\mathbb{R}$ such that $a \in I$, $b \in J$ and $I \times J \subseteq B((a, b), r)$. **(5 marks)**

3 (i) Define what it means for a function $F\colon U \to \mathbb{R}$ defined on an open set $U \subseteq \mathbb{R}^2$ to be differentiable at a point $(a, b) \in U$. **(3 marks)**

(ii) Show that $2\,|x|\,|y| \leqslant x^2 + y^2$ for all $x, y \in \mathbb{R}$. **(3 marks)**

(iii) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

(a) Calculate the partial derivatives of F at $(0, 0)$. **(2 marks)**

(b) Show that F is differentiable at $(0, 0)$. **(9 marks)**

(c) Use calculus to calculate the partial derivative $\dfrac{\partial F}{\partial x}(x, y)$ at each $(x, y) \neq (0, 0)$ and show that $\dfrac{\partial F}{\partial x}$ is continuous at $(0, 0)$. **(8 marks)**

4 (a) State carefully and in full the version of the Implicit Function Theorem which applies to maps with domain an open set $U \subseteq \mathbb{R}^3$ and target $\mathbb{R}^2$ and which gives conditions for a solution in terms of z. **(8 marks)**

Consider the simultaneous equations

$$(x^2 + y^2 + z^2 + 3)^2 = 16(y^2 + z^2), \qquad x^2 + y^2 + z^2 = 4. \qquad (*)$$

(b) Define a map $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ so that solutions of $(*)$ correspond to solving $\varphi(x, y, z) = (0, 0)$. **(2 marks)**

(c) Check that the two points $(\frac{\sqrt{15}}{4}, \frac{7}{4}, 0)$ and $(\frac{\sqrt{15}}{4}, 0, \frac{7}{4})$ are solutions of $(*)$. **(2 marks)**

(d) Apply the Implicit Function Theorem as you stated it in (a) to each of the points in (c).

At one of the two points, the Implicit Function Theorem may be applied and shows that it is possible to solve for x and y differentiably in terms of z. Determine which point this is.

Solve $(*)$ explicitly and, giving a simple sketch of the solutions, explain in terms of the solutions, or the diagram, why it is possible to solve for x and y differentiably in terms of z at one of these points and not the other. **(13 marks)**

5 Define $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\varphi(x, y, z) = (-x - y - z,\ xy + yz + zx,\ -xyz)$. You are given that φ is of class C^1.

(a) Calculate $D(\varphi)(x, y, z)$ and determine the rank of φ at each point of $\mathbb{R}^3$. **(9 marks)**

(b) Consider a cubic equation $t^3 + ut^2 + vt + w = 0$ for which there are three real roots x, y, z (two or three of which might be equal). Show that the roots satisfy $\varphi(x, y, z) = (u, v, w)$. **(4 marks)**

(c) State carefully and in full the version of the Local Diffeomorphism Theorem which applies to maps with domain and range which are subsets of $\mathbb{R}^3$. **(6 marks)**

(d) Apply the Local Diffeomorphism Theorem, as you stated it in (c), to φ from (a). Deduce that if the cubic equation $t^3 + u_0t^2 + v_0t + w_0 = 0$ has three distinct real roots x_0, y_0, z_0 then there are open intervals I containing u_0, J containing v_0 and K containing w_0, and a C^1 map $\psi\colon I \times J \times K \to \mathbb{R}^3$ such that for all $(u, v, w) \in I \times J \times K$ the roots of $t^3 + ut^2 + vt + w = 0$ are $\psi_1(u, v, w)$, $\psi_2(u, v, w)$ and $\psi_3(u, v, w)$. **(6 marks)**

End of Question Paper