The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2009-2010**

Mathematics (Numerical Methods and Vector Spaces) **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1. (i) Let A and B be the matrices

$$A=\begin{pmatrix}4 & -2 & 0 & 0\\ -1 & 4 & -1 & 0\\ 0 & -1 & 4 & -1\\ 0 & 0 & -1 & 4\end{pmatrix},\qquad B=\begin{pmatrix}56 & 30 & 8 & 2\\ 15 & 60 & 16 & 4\\ 4 & 16 & 56 & 14\\ 1 & 4 & 14 & 52\end{pmatrix}$$

Calculate AB and hence write down A^{-1}. **(*5 marks*)**

(ii) The heat equation $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$ is to be approximately solved in the region $\{(x,t): 0\le x\le 4,\, t\ge 0\}$, with initial condition $u(x,0)=x^2$ and boundary conditions $\dfrac{\partial u}{\partial x}(0,t)=0,\, u(4,t)=16$ for all t, using the Crank-Nicolson scheme

$$-ru_{i-1,\,j+1}+2(1+r)u_{i,\,j+1}-ru_{i+1,\,j+1}=ru_{i-1,\,j}+2(1-r)u_{i,\,j}+ru_{i+1,\,j}$$

Taking $h=k=1,\, r=k/h^2$, write out the equations which have to be solved to find $u(x,1)$ for $x=0,1,2,3$ and explain how to eliminate any fictitious values which arise.

Hence using the result of part (i) above find the values of

$u(0,1),\, u(1,1),\, u(2,1),\, u(3,1)$. **(*20 marks*)**

2. (i) Let $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -2 & 3 \end{pmatrix}$. Find the LU decomposition of A, where L is a lower triangular matrix with ones on the principal diagonal and U is an upper triangular matrix. **(*6 marks*)**

(ii) Verify that $L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{4} & \frac{3}{4} & 1 \end{pmatrix}$ and $U^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{8} & \frac{1}{18} \\ 0 & \frac{3}{8} & \frac{1}{6} \\ 0 & 0 & \frac{4}{9} \end{pmatrix}$ **(*2 marks*)**

(iii) Explain how you would use the result of part (ii) to find A^{-1}. Given that A^{-1} has the form $\frac{1}{18}\begin{pmatrix} 7 & 3 & 1 \\ a & b & c \\ 2 & 6 & 8 \end{pmatrix}$, find the values of a, b, c.

(*4 marks*)

(iv) The solution of the partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2$ is to be approximated in the square region $\{(x, y): 0 \le x \le 3, 0 \le y \le 3\}$. subject to the boundary conditions:

$$u(0, y) = y(3 - y)$$
$$u(x,0) = u(x,3) = 2x$$
$$\frac{\partial u}{\partial x}(3, y) = 2$$

Taking $h = k = 1$, complete the following grid by inserting the known values of u

$j \backslash i$	0	1	2	3	4
0					
1		u_1	u_2	u_3	u_4
2		u_1	u_2	u_3	u_4
3					

(*3 marks*)

(v) Write down equations relating the variables specified in part (iv) and, by eliminating any fictitious values, show that they are of the form $A\mathbf{u} = \mathbf{b}$ where A is the matrix in part (i) and $\mathbf{u}$ and $\mathbf{b}$ are column vectors. Hence solve the equations for the unknown values of $\mathbf{u}$.

(*10 marks*)

3. A cubic spline is to be fitted to the following set of data:

x	0	2	3
$f(x)$	20	120	80

with the additional constraints that the first derivative at $x = 0$ should be zero and the second derivative at $x = 3$ should be zero.

(i) Write down equations for the spline in each of the two intervals involved, using the conventional names for the unknown coefficients. ***(2 marks)***

(ii) Find the equations which must be satisfied by these coefficients so that the spline has the required properties. ***(8 marks)***

(iii) List the order in which the variables should be eliminated to provide a neat solution of the equations. ***(2 marks)***

(iv) Solve the equations to show that the spline has equation $f(x) = -\frac{47}{2}x^3 + 72x^2 + 20$ on the interval $[0,2]$, and find the corresponding equation on the interval $[2,3]$. ***(13 marks)***

4. (a) Check that the vectors

$$(1, 1, -2), (2, -1, 0) \text{ and } (1, 0, 2)$$

form a basis of $\mathbb{R}^3$ but are not an orthogonal set.
Find coordinates c_1, c_2 and c_3 of the vector

$$f = (1, 1, 1)$$

in this basis. **(*9 marks*)**

(b) Show that if $\{f_i\}_{i=1}^{N}$ is an orthonormal basis of the complex Hilbert space V, then the norm of the vector $f = \sum_{i=1}^{N} c_i f_i$ is

$$\|f\|^2 = \sum_i |c_i|^2 .$$

Give an expression for the inner product of the two vectors $f = \sum_{i=1}^{N} c_i f_i$ and $g = \sum_{i=1}^{N} d_i f_i$ in V in terms of their components with respect to the orthonormal basis $\{f_i\}$. **(*6 marks*)**

(c) Check that the vectors

$$f_1 = \frac{1}{\sqrt{2}}(\mathrm{j}, 0, 1), f_2 = (0, 1, 0), f_3 = \frac{1}{\sqrt{2}}(-\mathrm{j}, 0, 1)$$

form an orthonormal basis of the Hilbert space $\mathbb{C}^3$ and hence find the coordinates of the vector

$$g = (\mathrm{j}, 2\mathrm{j}+1, 2)$$

in this basis.

Find the norm of the vector g using the formula of part (b). **(*10 marks*)**

5. (a) If V is a Hilbert space with orthonormal basis $\{f_i\}_{i=-N}^{N}$ show that the minimum mean square approximation to the vector f in V, using the subset $\{f_i\}_{i=-M}^{M}$ where $M < N$, is given by

$$f_M = \sum_{i=-M}^{M} c_i f_i$$

where $\{c_i\}(-M \le i \le M)$ are the central $2M+1$ coordinates of f, i.e. $c_i = (f, f_i)$. **(*7 marks*)**

(b) The Hilbert space V of periodic signals of period T has the inner product

$$(f, g) = \frac{1}{T}\int_{-T/2}^{T/2} f(t) g^*(t)\,\mathrm{d}t\,.$$

Write down the associated norm and interpret it physically. **(*3 marks*)**

(c) You can assume that the periodic functions $f_n(t) = \mathrm{e}^{jnwt}$, where $w = 2p/T$ and $-\infty < n < \infty$, form an orthonormal basis for V.
Given the periodic function $f(t)$ defined by

$$f(t) = \begin{cases} 1 & 0 \le t < T/2 \\ 0 & T/2 \le t < T \end{cases}$$

find its coordinates c_n in the orthonormal basis $\{f_n\}$ using the formula $c_n = (f, f_n)$ and hence find the coefficients d_n in the minimum mean-square approximation

$$f_M(t) = \sum_{n=-M}^{M} d_n f_n$$

of $f(t)$. **(*15 marks*)**

6. (a) A random signal $f(t)$ is sampled at times $t = 0, T$ and $2T$ seconds, where $T = 1/2$, to produce the digital signal of length 3, $(f[0], f[1], f[2])$. The autocorrelation function, $R_f(t)$, of $f(t)$ is given by

$$R_f(t) = \frac{s^2}{1+|t|+2t^2}.$$

Write down the correlation matrix, R, of $f[n]$. Show that R is symmetric and that one of its eigenvalues is $\frac{3s^2}{4}$. Find the other eigenvalues. **(*13 marks*)**

(b) These digital signals are to be compressed using a single member of the Karhunen-Loève basis. Find this basis vector and determine the minimum square error associated with this compression. (Work to 4 decimal places.) **(*12 marks*)**

End of Question Paper