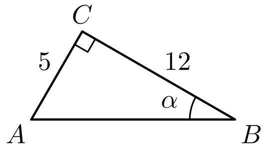
The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

FOUNDATION YEAR MATHEMATICS II **1 hour 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 For the triangle below, calculate $\sin\alpha$ and $\cot\alpha$.



(3 marks)

2 Draw a detailed sketch of the graph of $y = \cos x$ for $-180^\circ \leq x < 180^\circ$. Find all the values of θ in the range $-180^\circ \leq \theta < 180^\circ$ for which $\cos\theta = 0.25$. ***(4 marks)***

3 Points A and B lie on the circumference of a circle with centre O and radius r. Given that the arc AB has length π and the angle AOB is $\frac{\pi}{4}$ rad, find the radius r and the area of the triangle OAB. ***(4 marks)***

4 Find the equation of the line which is perpendicular to $2y - 3x = 2$ and passes through the point $(1, 1)$. ***(3 marks)***

5 Let $A = (0, y, 0)$ and $B = (0, 0, z)$ be points on the y and z-axes respectively, and let O be the origin. Given that $AB = \sqrt{10}$ and $OA + OB = 4$, find the possible values of y and z. ***(5 marks)***

6 The vectors **a**, **b**, **c** are given by

$$\mathbf{a} = (p, 0, -1), \quad \mathbf{b} = (2, q, 2), \quad \mathbf{c} = (0, r, r)$$

where p, q and r are unknown. Given that $3\mathbf{a}+\mathbf{b} = \mathbf{c}$, find p, q and r. *(4 marks)*

7 (i) State the geometric interpretations of

(a) the scalar product, and

(b) the vector product

taking care to explain any notation that you use. *(4 marks)*

(ii) Show that if **a** and **b** are perpendicular vectors then $\mathbf{a} \cdot \mathbf{b} = 0$. *(2 marks)*

8 Given vectors $\mathbf{a} = (3, 4, 0)$ and $\mathbf{b} = (3, 0, 4)$, calculate

(i) $\mathbf{a} - 2\mathbf{b}$,

(ii) $|2\mathbf{a} - \mathbf{b}|$,

(iii) the sum of unit vectors, $\hat{\mathbf{a}} + \hat{\mathbf{b}}$. *(6 marks)*

9 Let $A = (-1, 0, 0)$, $B = (2, 3, 2)$ and $C = (1, 1, 2)$ be points in 3-dimensions. Calculate

(i) $\overrightarrow{BA}$ and $\overrightarrow{BC}$ in component form, *(2 marks)*

(ii) $\overrightarrow{BA} \cdot \overrightarrow{BC}$, and so the angle ABC, *(3 marks)*

(iii) $\overrightarrow{BA} \times \overrightarrow{BC}$, and so the area of the triangle ABC. *(3 marks)*

10 Find the vector equation of the line L_1 which passes through the point $A = (2, 1, 6)$ in the direction of $\mathbf{d} = (1, 1, 1)$. Show that L_1 does not intersect the line L_2 with vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 10 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

(5 marks)

Find the value of a for which the line L_3 with equation

$$\mathbf{r} = \begin{pmatrix} a \\ 0 \\ 6 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

passes through the point A. For this value of a, do L_1 and L_3 intersect?

(2 marks)

End of Question Paper

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SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2010–2011**

FOUNDATION YEAR MATHEMATICS II — **3 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) A tightrope is drawn between two posts (post A and post B) of height h a distance d apart, in metres. A tightrope walker stands at the top of post A and observes the bottom of post B at an angle θ below the horizontal. After walking 10m along the length of the rope, the walker observes the bottom of post B at an angle α below the horizontal.

(a) Express $\tan\theta$ and $\tan\alpha$ in terms of h and d, and hence show that

$$d = \frac{10\tan\alpha}{\tan\alpha - \tan\theta}.$$

(4 marks)

(b) Given that $\theta = 30^\circ$ and $\alpha = 45^\circ$, find the distance d and the height h, correct to 2 significant figures. ***(2 marks)***

(ii) An aircraft takes off from a point O on a runway and travels with a constant velocity of $\mathbf{v}$ ms^{-1}. Relative to axes $Oxyz$, $\mathbf{v}$ is given by

$$\mathbf{v} = (30, 40, 10),$$

where the axis Oz is vertically upwards.

(a) Calculate the speed of the aircraft. ***(2 marks)***

(b) After 30 seconds, the aircraft is vertically above a control tower. Find the height of the aircraft when it passes over the control tower, the distance of the control tower from O and the angle the plane's path makes with the ground. ***(5 marks)***

(You should assume that the ground is perfectly flat.)

(iii) In the triangle OAB the midpoint of AB is denoted by M. Given that $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B relative to O, draw a diagram to illustrate the situation and express $\overrightarrow{AB}$, $\overrightarrow{OM}$ and $\overrightarrow{AB} \cdot \overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Hence show that if $\overrightarrow{OM}$ and $\overrightarrow{AB}$ are perpendicular, then the triangle must be isosceles. ***(7 marks)***

2 (i) You are given that

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B.\end{aligned}$$

(a) Using the addition formulae given above, show that

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}.$$ *(6 marks)*

(b) Let $t = \tan 67.5°$. Using part (a), show that

$$t^2 - 2t - 1 = 0.$$

Hence find a precise expression for $\tan 67.5°$, showing your workings. *(4 marks)*

(ii) Expand and simplify $(1+2x)^5$. *(3 marks)*

(iii) For each of the following, state whether or not it is a geometric progression. If it is, state also its common ratio and find the sum to infinity if it exists.

(a) $3, -3, -9, -27, -81, \ldots$

(b) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$

(c) $1 - 2x + 4x^2 - 8x^3 + 16x^4 + \ldots$ (where $|x| < \frac{1}{2}$).

(7 marks)

3 (i) Complex numbers z_1 and z_2 are given by $z_1 = 3+i$ and $z_2 = 2-i$. Evaluate z_1z_2 and $\frac{z_1}{z_2}$, giving your answers in the form $a+ib$ where a and b are real numbers. *(5 marks)*

(ii) Let $z = 3+4i$.

(a) Draw z and $\overline{z}$ on an argand diagram. *(2 marks)*

(b) Calculate $|z|$ and explain, with reference to your diagram, why $|\overline{z}|$ has the same value. *(3 marks)*

(c) Calculate the principal argument, $\arg(z)$, and hence write down the value of $\arg\overline{z}$. *(3 marks)*

(d) Describe the path on the argand diagram that the complex number w traces if $|w-z| = 5$, and draw this path on an argand diagram. *(3 marks)*

(iii) Find the radii and co-ordinates of the centre of the circles below.

(a) $x^2 + y^2 - 4x + 2y - 48 = 0$.

(b) $2x^2 + 2y^2 - 8y - 1 = 0$.

(4 marks)

4 (i) Use the iterative formula

$$x_{r+1} = 2 + \frac{4}{x_r}$$

with $x_1 = 3$ to find $x_2,\ x_3$ and x_4 correct to 3 significant figures. Find, in its simplest form, the equation that is solved by this iterative formula and hence find an exact expression for the solution that was approximated by x_4. ***(8 marks)***

(ii) Evaluate

$$\int_0^{\frac{\pi}{4}} \sin x \ dx$$

working to three decimal places

(a) by ordinary integration;

(b) using the trapezium rule with four strips;

(c) by Simpson's rule with four strips. ***(12 marks)***

5 (i) Simplify $\dfrac{(2n)!(n-1)!}{(2n-1)!n!}$. ***(3 marks)***

(ii) Express the recurring decimal $0.17\dot{3}$ as a fraction. ***(5 marks)***

(iii) The size S of a population of a city at time t (in years) satisfies the differential equation

$$\frac{dS}{dt} = kS,$$

where k is a constant.

(a) By separating the variables, integrate this equation to find S in terms of t and k. ***(5 marks)***

(b) At $t = 0$ (in 2010), the population is 100,000. In 2020, the population is 135,000. Approximately how many people will live in the city in 2050? ***(7 marks)***

End of Question Paper