Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010–11**

MAS140(1) Mathematics (Chemical)
MAS143 Civil Engineering Mathematics I
MAS149 Essential Mathematical Techniques A

2 hours

*Attempt **ALL** questions.*

Each question in Section A carries 4 marks, each question in Section B carries 8 marks.

Section A

A1 Use the binomial theorem to evaluate $\lim_{x\to 0} \frac{(8-3x)^{1/3}-2}{x}$.

A2 Express the function e^x as the sum of an odd function and an even function.

A3 On the assumption that x and $f(x)$ are real, find the domain and range of the function $f(x) = \sqrt{x-1}\sin x$.

A4 If $f(x) = \ln(1+x)$, find $f^{-1}(x)$.

A5 Sketch the function $f(x) = |x+1| + |x-1|$, $(-2 \leq x \leq 2)$.

A6 Differentiate the function $g(x) = 2^x x^2$.

A7 Find the minimum value of the function $h(x) = x\ln x - 2x$.

A8 If $f(x,y) = xy^2\cos(x^2y)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

A9 Evaluate $\lim_{x\to 0} \frac{x-\tan x}{x-\sin x}$.

A10 Find the 4 complex numbers for which $z^2 = \bar{z}$.

A11 Plot the 3 values of $i^{1/3}$ on a sketch of the Argand diagram.

A12 If $\mathbf{a} = (5, 2, -3)$ and $\mathbf{b} = (-2, 2, 3)$, find $\mathbf{a}.\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

A13 Find the angle between the vectors $\mathbf{a} = (2, -2, -3)$ and $\mathbf{b} = (3, -2, 5)$.

Section B

B1 Find all maxima and minima of the function

$$f(x) = \frac{2x^2 + 4x - 1}{x^2 + 1}.$$

Sketch this function.

B2 Write down the first 4 terms of the Taylor expansion of $f(x)$ about $x = a$. Find these terms for the expansion of $f(x) = \sqrt{x}$ about $x = 9$.

B3 Show that if $r = \sqrt{x^2 + y^2}$ and $f(x, y) = F(r)$ then

$$\frac{\partial f}{\partial x} = F'(r)\frac{x}{r}$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = F''(r) + \frac{F'(r)}{r}.$$

B4 Define $\cosh x$ and from your definition show that

$$\cosh 3x = 4\cosh^3 x - 3\cosh x.$$

B5 Given that x and y are real, find all values of x and y for which

$$\frac{3x - y + i}{x + 2i} + \frac{y + 2i}{1 + i} = 2.$$

B6 The position vector of a particle at time $t > 0$ is given by

$$\mathbf{r} = (t/2, 2\sqrt{t}, \ln t).$$

Find the velocity ($\mathbf{v}$), speed (v) and acceleration ($\mathbf{a}$).
Verify that

$$t^2\mathbf{v}.\mathbf{a} = -v.$$

End of Question Paper

Formula Sheet for MAS140/143/149

All of the following results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$\sin^2\theta + \cos^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \mathrm{cosec}^2\theta$
$\cos^2\theta = (1 + \cos 2\theta)/2$
$\sin^2\theta = (1 - \cos 2\theta)/2$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$ where
$R = \sqrt{a^2 + b^2}$, $\cos\alpha = a/R$ and $\sin\alpha = b/R$

Hyperbolic Functions

$\cosh^2 x - \sinh^2 x = 1$
$\mathrm{sech}^2 x + \tanh^2 x = 1$
$\cosh^2 x + \sinh^2 x = \cosh 2x$
$2\sinh x \cosh x = \sinh 2x$
$\cosh^2 x = (1 + \cosh 2x)/2$
$\sinh^2 x = -(1 - \cosh 2x)/2$

Series

Sum of an arithmetic series:

$$\frac{\textit{first term} \ + \ \textit{last term}}{2} \times (\textit{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r}x^r + \ldots$

where $\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \le 1)$$

Differentiation

Function	Derivative
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$

N.B. It is assumed that x takes only values for which the function is defined.

Integration

Function	Integral
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$
$\text{cosec}\, x$	$\ln\tan\left(\frac{x}{2}\right)$ or $\ln(\text{cosec}\, x - \cot x)$
$\sec x$	$\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$ or $\ln(\sec x + \tan x)$
$\text{cosech}\, x$	$\ln\tanh\left(\frac{x}{2}\right)$

If $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{dx}{dt} = \frac{2}{1+t^2}$.

Integration-by-parts

$$\int_a^b uV\,dx = [u \times (integral\ of\ V)]_a^b - \int_a^b (integral\ of\ V) \times \frac{du}{dx}dx$$

or $\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

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MAS140

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2010-2011**

Mathematics (Chemical) **2 hours**

*Marks will be awarded to **all** questions. The marks assigned to each question are shown in italics.*

1 Evaluate the definite integral $\displaystyle\int_0^{\cosh^{-1}(\sqrt{3})} \frac{\sinh\theta}{\cosh^2\theta+1}\,d\theta.$ *(8 marks)*

2 Evaluate the indefinite integral $\displaystyle\int \frac{\ln x}{x^2}\,dx.$ *(8 marks)*

3 Find the determinant of

$$A = \begin{bmatrix} 1 & 0 & 6 & 2 \\ 3 & 4 & 4 & 1 \\ -1 & 0 & 2 & -1 \\ 3 & 3 & 5 & 1 \end{bmatrix}.$$

(9 marks)

4 Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix},$$

and for each eigenvalue find a corresponding eigenvector. *(8 marks)*

5 Find the specific solution of the differential equation

$$\frac{1}{y+3}dy = \left[\sin x + (x+1)^2\right] dx$$

such that $y = -2$ at $x = 0$. Your answer should be in the form $y = f(x)$.

(9 marks)

6 Find the general solution of the differential equation

$$2x\frac{dy}{dx} + 2y = x - 2xy.$$

Your answer should be in the form $y = f(x)$. *(8 marks)*

7 Let

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 9 & 1 \\ 2 & 5 & 1 \end{bmatrix}.$$

(i) Compute $|A|$. Why does your answer show that A is invertible?

(5 marks)

(ii) Find A^{-1}. *(7 marks)*

(iii) Solve the system of linear equations

$$\begin{aligned} 2x + 4z &= 1 \\ 3x + 9y + z &= 1 \\ 2x + 5y + z &= 1. \end{aligned}$$

(3 marks)

8 Evaluate the indefinite integral

$$\int \frac{5x^4 - 3x^3 + 8x^2 - 8x + 3}{x^5 - 2x^4 + 3x^3}\, dx.$$

(17 marks)

9 Find the solution of the differential equation

$$y'' - 2y' + 2y = e^x(\sin 2x - 1)$$

such that $y = 1, y' = -1$ when $x = 0$. *(18 marks)*

End of Question Paper