The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2010–11

Linear Mathematics for Applications — **2 hours**

Attempt all questions. The allocation of marks is shown in brackets.

1 Let

$$A := \begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 1 \\ -1 & 1 & -1 & 0 & -2 \end{pmatrix}.$$

(i) Transform the matrix A by a sequence of elementary row operations to a matrix in reduced row echelon form. ***(5 marks)***

(ii) What is the rank of A ? ***(1 mark)***

(iii) Determine the general solution of the system of linear equations $AX = 0$, where $X := \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}$, and write your general solution in (column) vector form. ***(5 marks)***

(iv) Let

$$\mathcal{N}_A := \left\{ v \in \mathbb{R}^5 \mid Av = 0 \right\}.$$

(a) Show that the set $\mathcal{N}_A$ is a subspace of $\mathbb{R}^5$.

(b) Find a basis for the subspace $\mathcal{N}_A$.

(c) What is the dimension of the space $\mathcal{N}_A$?

(7 marks)

(v) Let $v_1 = (1, 2, -1)^T$, $v_2 = (-1, -2, 1)^T$, $v_3 = (-1, -1, -1)^T$. Let $W = \mathrm{Sp}\{v_1, v_2, v_3\}$.

(a) What is the dimension of the subspace W ? Justify your answer.

(b) Find a basis of W, and justify your answer.

(c) Let $v_4 = (2, 3, 0)^T$ and $v_5 = (1, 1, -2)^T$. Let $V = \mathrm{Sp}\{v_1, v_2, v_3, v_4, v_5\}$. Find a basis of V.

(7 marks)

2 Let

$$A := \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix}$$

(i) Find the eigenvalues of A, and for each eigenvalue a corresponding eigenvector. ***(9 marks)***

(ii) Express the vectors $(1, 0)^T$ and $(0, 1)^T$ as linear combinations of the eigenvectors found in part (i). ***(5 marks)***

(iii) The weather in Sheffield is either rainy or dry. As a result of extensive record keeping, it has been determined that the probability of a rainy day following a dry day is 1/3, and the probability of a rainy day following a rainy day is 1/2.

If it is dry today, find the probability it is raining in five days time.

If it is raining today, find the probability it is dry in five days time.

An answer to one decimal place is sufficient. ***(11 marks)***

3 (i) State whether or not each of the following statements is true in general. You do not need to justify your answer.

(a) Let A be an $n \times n$ symmetric matrix. Then $\mathbb{R}^n$ has a basis made up of eigenvectors of A.

(b) Let $v_1, \ldots, v_k$ be linearly independent vectors in $\mathbb{R}^n$. Then $k < n$.

(c) Let A and B be matrices of the same size, where $A \sim B$. Then A and B have the same column space.

(6 marks)

(ii) For each of the following subsets L_i ($i = 1, 2, 3, 4, 5$) of $\mathbb{R}^3$, determine, with justification, whether L_i is a subspace of $\mathbb{R}^3$. If L_i is a subspace, find its dimension.

(a) $L_1 := \{(x, y, z)^T \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$;

(b) $L_2 := \{(x, y, z)^T \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0\}$;

(c) $L_3 := \{(x, y, z)^T \in \mathbb{R}^3 \mid xyz = 0\}$;

(d) $L_4 := \{(x, y, z)^T \in \mathbb{R}^3 \mid x + y + z = 0\}$;

(e) $L_5 := \{(x, y, z)^T \in \mathbb{R}^3 \mid x^3 + y^3 + z^3 = 0\}$.

(12 marks)

(iii) Calculate the determinants of the following matrices.

(a)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 4 & 3 & 2 & 0 \\ 5 & 4 & 3 & -2 \end{pmatrix},$$

(b)

$$B = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix},$$

(c) $C = AB$.

(7 marks)

4 Let

$$A := \begin{pmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{pmatrix}.$$

(i) Find the eigenvalues of the matrix A, and for each eigenvalue write down a corresponding eigenvector. ***(10 marks)***

(ii) Find the general solution of the system of linear differential equations

$$\begin{aligned} y_1' &= -2y_3, \\ y_2' &= -2y_2, \\ y_3' &= -2y_1 + 3y_3. \end{aligned}$$

(4 marks)

(iii) Write down an orthogonal matrix P and diagonal matrix D such that $P^T AP = D$. ***(5 marks)***

(iv) Let $Q(x, y, z)$ be the real quadratic form given by

$$Q(x, y, z) = -2y^2 + 3z^2 - 4xz.$$

(a) Determine the rank and signature of the quadratic form $Q(x, y, z)$.

(b) Determine the nature of the quadric surface in $\mathbb{R}^3$ whose equation is $Q(x, y, z) = 1$.

(c) Determine the maximum and minimum values in the set

$$K := \left\{Q(a, b, c) : a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 + c^2 = 1\right\}.$$

(6 marks)

End of Question Paper