The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2010-11**

Advanced Calculus 2 **hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) The random variables X and Y have joint density function

$$f_{X,Y}(x, y) = \begin{cases} e^{-xy^2}, & \text{if } x \geqslant 0 \text{ and } y \geqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the probability $P\left(XY^2 \leqslant 1\right)$. ***(9 marks)***

(ii) Let $\omega = (4x^3 + 2xy^2)\,dx + (4y^3 + 2yx^2)\,dy$.

(a) Show, *without* finding a potential function, that ω is an exact differential.

(b) Now find a potential function f for ω.

(c) Evaluate the line integral $\int_\gamma \omega$, where $\gamma : x = \cos t, y = t^2, \; 0 \leqslant t \leqslant \pi/2$. ***(12 marks)***

(iii) A function F is defined by the formula

$$F(x) = \int_0^{x^2} \sin(t^2)\,dt.$$

Write down an expression for the derivative $\frac{dF}{dx}$. (Do not attempt to calculate $F(x)$ by evaluating the integral.) ***(4 marks)***

2 (i) State Green's Theorem, being careful to include any conditions needed for its validity. Hence evaluate

$$\int_C (xy^3 + y\cos(xy))dx + (2x^2y^2 + x\cos(xy))dy,$$

where C is the triangular path with vertices $(0,0)$, $(1,1)$ and $(0,1)$, described in the anticlockwise direction. ***(15 marks)***

(ii) Let C be the curve parametrised by

$$x = 4\sin(t)\cos(t),\ y = 2\sin^2 t,\ 0 \leqslant t \leqslant \pi.$$

Calculate the area of the region D enclosed by C (You may assume that C satisfies the hypothesis of Green's Theorem). ***(10 marks)***

3 Let f be the periodic function with period 2π such that

$$f(x) = \begin{cases} 2x, & \text{for } 0 \leqslant x < \pi; \\ -2x, & \text{for } -\pi \leqslant x < 0. \end{cases}$$

(i) Sketch the graph of the function f. Calculate all the coefficients in the Fourier series for f. ***(13 marks)***

(ii) What can you deduce by comparing the results of plugging in $x = 0$ to $f(x)$ and to its Fourier series? ***(4 marks)***

(iii) Sketch the graph of the derivative of f (where it is defined). Assuming that the Fourier series for f may be differentiated term-by-term to get the Fourier series for the derivative f' of f, what can you deduce by plugging in $x = \pi/2$? ***(8 marks)***

4 (i) If X is a random variable, define the variance σ^2 of X and the probability generating function $G_X(s)$. In the case that X is a discrete random variable taking only the values $0, 1, 2, \ldots$, show that the mean of X is given by $E[X] = G'_X(1)$ and the mean of $X(X-1)$ is given by $E[X(X-1)] = G''_X(1)$. Deduce that $E[X^2] = G''_X(1) + G'_X(1)$.

(8 marks)

Suppose that X is the Binomial distribution with parameters n and p where $0 \leqslant p \leqslant 1$. Given that the probability generating function is $G_X(s) = (q+ps)^n$ where $q = 1-p$,

(a) Show that the mean $\mu = E[X]$ of X is equal to np.

(b) Show that the variance σ^2 of X is equal to npq. ***(9 marks)***

(ii) Recall the Fourier transform $\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$ (when it exists), and the Fourier inversion formula

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(x)e^{itx}\,dx$$

(valid under certain conditions). Let

$$u(t) = \begin{cases} t, & \text{if } -1 \leqslant t \leqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\hat{u}(s) = 2i\left(\frac{\cos(s)}{s} - \frac{\sin(s)}{s^2}\right)$. ***(8 marks)***

5 (i) Find and classify the critical points of the function

$$f(x, y) = -y^3 + x^2 - 2xy - 2x + 2y.$$

(13 marks)

(ii) Find the points of the curve

$$x^2 + y^2 - 6x - 2y + 9 = 0$$

closest to the origin. ***(12 marks)***

End of Question Paper